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SECOND ORDER NECESSARY CONDITIONS FOR

GENERAL PROBLEMS WITH STATE INEQUALITY CONSTRAINTS

by

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ABSTRACT:

This paper is a sequel to an article by the author, concerned with a certain canonical problem in optimal control involving constraints of the type $\psi^{\alpha}(t,x) \leq 0 \quad \alpha = 1,\ldots,m$. In that article a set of second order conditions necessary for a solution arc was obtained. In this paper those results are extended to a general control problem involving the above type of constraints.

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1. Introduction

Reference [4] is concerned with proving a set of second order necessary conditions for a basic problem in optimal control involving state inequality constraints. We will henceforth refer to that problem as the canonical problem. In this paper we will extend those results by proving a set of second order necessary conditions for problems of increasing complexity, building up to the general problem stated next:

Let C be the class of arcs

$$a: \quad x^i(t) \qquad u^k(t) \qquad b^\sigma \qquad t^0 \le t \le t^1$$

$$i = 1,\dots,N \qquad k = 1,\dots,K \qquad \sigma = 1,\dots,r$$

whose elements $(t, x(t), u(t), b)$ lie in a region R in $txub$ space and which in addition have $u(t)$ piecewise continuous and satisfy the constraints

$$\dot{x}^i = f^i(t,x,u,b) \qquad i = 1,\dots,N$$

$$\psi^\alpha(t,x,b) \le 0 \quad \alpha = 1,\dots,m' \qquad \psi^\alpha(t,x,b) = 0 \quad \alpha = m'+1,\dots,m$$

$$\theta^\eta(t,x,u,b) \le 0 \quad \eta = 1,\dots,L' \qquad \theta^\eta(t,x,u,b) = 0 \quad \eta = L'+1,\dots,L$$

$$I_\gamma(a) \le 0 \quad \gamma = 1,\dots,p' \qquad I_\gamma(a) = 0 \quad \gamma = p'+1,\dots,p$$

$$x^i(t^s) = X^{is}(b) \qquad i = 1,\dots,N \qquad s = 0,1$$

where

$$I_\gamma(a) = g_\gamma(b) + \int_{t^0}^{t^1} L_\gamma(t,x(t),u(t),b)dt \qquad \gamma = 1,\dots,p$$

It is desired to minimize the integral

$$I_0(a) = g_0(b) + \int_{t^0}^{t^1} L_0(t,x(t),u(t),b)dt$$

on the class C.

The proof will be based upon our ability at each step of added complexity, to reduce the more complicated problem to one for which the required results have already been established. This will be done by using the transformations of [2] and [1]. In this procedure we will generally refer to the particular problem under considerations as the original problem and to the result of transforming it, as the transformed problem. Since it is too lengthy to repeat the transformations of [1] and [2] here, familiarity with those papers will be assumed.(1)

Hestenes [3] treated a special case of the general problem in which the endpoints were fixed and only isoperimetric constraints appeared. Pennisi [5] in doing a sufficiency proof for a different version of the fixed endpoint problem in which only differential constraints appeared, stated a second order condition for that problem. For both simpler versions the results obtained here reduce to the results obtained or used there.

2. Statement of the Canonical Problem and Results - Since we will have frequent need to refer to the canonical problem and the associated results obtained in [4], we repeat here the statement of that problem and those results. Let C be the class of arcs

$$a: \quad x^i(t) \qquad u^k(t) \qquad b^\sigma \qquad t^o \le t \le t^1$$

$$i = 1,\dots,N \qquad k = 1,\dots,K \qquad \sigma = 1,\dots,r$$

(1) Unless otherwise specified, all terms used here will have the meaning defined in [1] or [2] as the case may be.

whose elements[1] $(t,x(t),u(t))$ and b lie respectively in open sets R in $t \times u$ space[2] and B in b space and which in addition have $u(t)$ piecewise continuous. The terms x^i are called state variables. The terms u^k, b^σ are called control variables and control parameters respectively. We require these arcs to satisfy the constraints.

(1a) $$\dot{x}^i = f^i(t,x,u) \qquad i = 1,\dots,N$$

(1b) $$\psi^\alpha(t,x) \le 0 \qquad \alpha = 1,\dots,m$$

(1c) $$I_\gamma(a) \le 0 \qquad \gamma = 1,\dots,p' \qquad I_\gamma(a) = 0 \qquad \gamma = p' + 1,\dots,p$$

(1d) $$x^i(t^s) = X^{is}(b) \qquad s = 0,1 \qquad i = 1,\dots,N$$

where:

(1e) $$I_\gamma(a) = g_\gamma(b) + \int_{t^0}^{t^1} L_\gamma(t,x(t),u(t))dt \qquad \gamma = 1,\dots,p$$

It is desired to minimize the integral

(2) $$I_0(a) = g_0(b) + \int_{t^0}^{t^1} L_0(t,x(t),u(t))dt$$

on the class C.

(1) Unless otherwise noted, the indices i, k, σ, α, γ will have the respective ranges $i = 1,\dots,N$; $k = 1,\dots,K$; $\sigma = 1,\dots,r$; $\alpha = 1,\dots,m$; $\gamma = 1,\dots,p$.

(2) We will often denote a vector by the same symbol as used for its components but with the component index removed, e.g., the vector with components $(x^1,\dots,x^N)$ is called x .

We suppose that the arc

(3) $a_0 \qquad x_0(t) \qquad u_0(t) \qquad b^0 \qquad t^0 \le t \le t^1$

is a solution to our problem. In addition, we set[(1)] $\phi^\alpha = \psi^\alpha_t + \psi^\alpha_{x^i} f^i$

and assume the following continuity properties for our functions:

i) ψ^α are of class C^3 and C^1 respectively with respect to x and t and ψ^α_t is of Class C^2 with respect to x,

ii) f^i, L_γ, g_γ, X^{ic} are of class C^2.

iii) the function $u_0(t)$ is continuous on $[t^0,t^1]$

The set R_0 is defined as the set of points $(t\ x\ u)$ in R satisfying

(4a) $$\psi^\alpha \le 0$$

(4b) $$\phi^\alpha \ge 0 \quad \forall\alpha \quad \text{with} \quad \psi^\alpha = 0 \qquad \text{or} \qquad \phi^\alpha \le 0 \quad \forall\alpha \quad \text{with} \quad \psi^\alpha = 0$$

$$\alpha = 1, \ldots, m$$

Let $\bar{D}$ be the set of points $(t,x_0(t),u)$ in R_0 with $u = u_0(t)$ or for arbitrary u, with t interior to an interval of continuity of $u_0(t)$. Then we assume that the matrix

(5) $$(\phi^\alpha_{u^k}) \qquad \alpha = 1, \ldots, m$$

has rank m on $\bar{D}$.

In [4] a basic second order theorem is proven in the case that the above problem is normal. For convenience, we repeat here the definition of normality

(1) Unless otherwise noted, repeated indices will be summed. For a function $M(t,x,u,b)$ the notations M_{x^i}, M_{u^k}, M_{b^σ}, M_t denote first partial derivatives with respect to the indicated variables.

(12b) $$\delta_\omega \psi^\alpha(t) = 0 \text{ on a neighborhood of } S^\alpha \quad \alpha = 1,\ldots, m$$

in addition to providing that the matrix

(12c) $$(J'_\rho(\mathcal{a}_0, \delta_\omega \mathcal{a})) \quad \rho = 1, \ldots, p + 2N \quad \rho \neq \gamma_k \quad \omega = 1, \ldots, \Omega$$

is non-singular.

Next define the class Y of admissible variations

$$\mathcal{y}: \qquad y^i(t) \qquad v^k(t) \qquad z^\sigma \qquad t^0 \leq t \leq t^1$$

of $\mathcal{a}_0$ which in addition satisfy(1)

(13a) $$\delta_{\mathcal{y}}\psi^\alpha(t^c) = 0 \text{ if } \mu_\alpha(t^c) \neq 0 \quad c = 0,1$$

(13b) $$\delta_{\mathcal{y}}\psi^\alpha(\bar{t}) = 0 \text{ if } \mu_\alpha(t) \text{ is not constant on a neighborhood of } \bar{t}$$

(13c) $$\delta_{\mathcal{y}}\psi^\alpha(t^0) = 0 \text{ if } \psi^\alpha(t^0) = 0 \qquad \alpha = 1,\ldots,m$$

(13d) $$J'_\gamma(\mathcal{a}_0, \mathcal{y}) = 0 \text{ if } \lambda_\gamma \neq 0 \quad 1 \leq \gamma \leq p'$$

(13e) $$J'_\gamma(\mathcal{a}_0, \mathcal{y}) \leq 0 \text{ if } \lambda_\gamma = 0 \quad \gamma \neq \gamma_k \quad 1 \leq \gamma \leq p'$$

(13f) $$J'_\rho(\mathcal{a}_0, \mathcal{y}) = 0 \quad p' < \rho \leq p + 2N$$

where γ_k are the indices introduced in (11) and where $\mu_\alpha(t), \lambda_\gamma$ are multipliers of Theorem 6.1 of [1].

(1) The notation $\delta_{\mathcal{y}}\psi^\alpha(t)$ denotes the variation in the function $\psi^\alpha(t)$ due to $\mathcal{y}$, similar notations will also be used for other functions and variations.

Next we list the functions H and G from Theorem 6.1 of [1]

$$H = p_i(t)f^i - \lambda_0 L_0 - \lambda_\gamma L_\gamma - \mu_\alpha(t)\phi^\alpha$$

$$G = \lambda_0 g_0 + \lambda_\gamma g_\gamma + \lambda_{p+i} X^{io}$$

$$i = 1,\dots,N \qquad \gamma = 1,\dots,p, \qquad \alpha = 1,\dots,m$$

where $p_i(t)$, $\lambda_\rho, \mu_\alpha(t)$ are multipliers from the above listed theorem.

With this introduction, then the result proven in [4] is Theorem 3.1 and is repeated here as follows:

Theorem 2.1 Let a_0 be a normal solution to the canonical problem. Then the multipliers $\lambda_\rho, \mu_\alpha(t), p_i(t), K^\alpha$, $\rho = 0,1,\dots,p+N$; $\alpha = 1,\dots,m$; $i = 1,\dots,N$ of Theorem 6.1 of [1] satisfy: i) $\lambda_0 \neq 0$ and ii) with $\lambda_0 = 1$, these multipliers are unique. Furthermore with G and H as the functions listed above then for each variation y in the class Y , we have

$$(14) \qquad 0 \le \Big[\big(p_i(t^c)\, X^{ic}_{b^\sigma b^\tau}\big)\Big|_{c=0}^{c=1} + G_{b^\sigma b^\tau} + K^\alpha \psi^\alpha_{\dot{x}^i \dot{x}^j}(t^0) X^{i0}_{b^\sigma} X^{j0}_{b^\tau}\Big] z^\sigma z^\tau$$

$$- \int_{t^0}^{t^1} \big[H_{x^i x^j} y^i y^j + 2H_{x^i u^k} y^i v^k + H_{u^h u^k} v^h v^k\big]dt$$

$$i,j = 1,\dots,N \qquad h,k = 1,\dots,K \qquad \sigma,\tau = 1,\dots,r$$

3. A Problem with a Relaxed Condition on the Rank of the Matrix $(\phi^{\alpha}_{u^k})$

As a first step in generalizing the previous result, consider the problem of section 2 of [1] as follows:

We are concerned with arcs

$$a: \qquad x^i(t) \quad u^k(t) \quad b^{\sigma} \qquad t^0 \le t \le t^1$$

$$i = 1,\ldots,N, \qquad k = 1,\ldots,K, \qquad \sigma = 1,\ldots,r$$

which have points $(t,x(t),u(t))$ in a region R in txu space, b in a region B in b space and $u(t)$ piecewise continuous. It is desired to minimize the integral

$$I_0(a) = g_0(b) + \int_{t^0}^{t^1} L_0(t,x(t),u(t))dt \tag{15}$$

in the class C of arcs which satisfy

$$\dot{x}^i = f^i(t,x,u) \qquad i = 1,\ldots,N$$

$$\psi^{\alpha}(t,x) \le 0 \qquad \alpha = 1,\ldots,m \tag{16}$$

$$I_{\gamma}(a) \le 0 \qquad \gamma = 1,\ldots,p' \qquad I_{\gamma}(a) = 0 \qquad \gamma = p'+1,\ldots,p$$

$$x^i(t^s) = X^{is}(b) \qquad s = 0,1 \qquad i = 1,\ldots,N$$

where $I_{\gamma}(a) = g_{\gamma}(b) + \int_{t^0}^{t^1} L_{\gamma}(t,x(t),u(t))dt \qquad \gamma = 1,\ldots,p,$

Now, assume that the arc

$$a_0: \quad x_0(t), \quad u_0(t), \quad b_0 \qquad t^0 \le t \le t^1 \tag{17a}$$

is a solution to the problem and define the set R_0 as the set of points (txu) in R satisfying

(17b) $$\psi^\alpha(t,x) \le 0$$

$$\phi^\alpha(t,x,u) \ge 0 \quad \forall\alpha \ \text{with} \ \psi^\alpha = 0 \quad \text{or} \quad \phi^\alpha(t,x,u) \le 0 \quad \forall\alpha \ \text{with} \ \psi^\alpha = 0$$

$\alpha = 1,\ldots,m$

Also define the set R_1 as the set of points $(t,x_0(t),u)$ in R_0. The point of generalization from the canonical problem is that we now relax the assumption involving the condition (5) and instead assume that the matrix[1]

(17c) $$(\phi^\alpha_{u^k} \qquad \delta_{\alpha.\beta}\psi^\beta) \qquad \beta,\alpha = 1,\ldots,m \qquad k = 1,\ldots,K$$

has rank m on R_1

Further details of this problem definition may be obtained from [1], however for our purposes, the present definition is sufficient.

The only modifications which we make to the problem of section 2 of [1] are as follows: (i) the control $u_0(t)$ along a_0 is continuous; (ii) the functions ψ^α are of class C^3 on R while the functions f^i, L_γ, g_γ, X^{is} are of class C^2 on R or B as the case may be.

4. Normality and the Second Order Theorem for the Problem of Section 3

Corresponding to the definition of normality for the canonical problem we will define normality for the present problem. Analagous to section 2, we let δa denote the variation about the arc a_0 of (17)

(1) The notation δ with double subscript will always denote the Kroncker Delta unless otherwise specified.

(18) $\qquad \delta a: \quad \delta x(t), \quad \delta u(t) \quad \delta b \qquad t^0 \le t \le t^1$

Next, define the functionals of an arc a as

$$J_\gamma(a) = \quad I_\gamma(a) \qquad \gamma = 0,1,\ldots,p$$

(19)
$$J_{p+i}(a) = x^i(t^1) - X^{i1}(b)$$

$$J_{p+N+i}(a) = X^{i0}(b) - x^i(t^0) \qquad i = 1,\ldots,N$$

and call the expressions,

(20)
$$J'_\gamma(a_0,\delta a) = I'_\gamma(a_0,\delta a)$$

$$J'_{p+i}(a_0,\delta a) = \delta x^i(t^1) - \delta X^{i1}$$

$$J'_{p+N+i}(a_0,\delta a) = \delta X^{i0} - \delta x^i(t^0)$$

the variations of the functionals (19) about the arc a_0 due to the variation δa.

Next, with

(21a)
$$\gamma_k \qquad k = 1,\ldots,\bar{p}$$

as the indices for which

(21b)
$$I_\gamma(a_0) < 0 \qquad 1 \le \gamma \le p'$$

we see by Theorem 3.2 of [1] that

(21c)
$$\lambda_{\gamma_k} = 0 \qquad k = 1,\ldots,\bar{p}$$

With $\mu_\alpha(t)$, $\lambda_{p+N+\alpha}$ as the terms of that theorem, we define a_0 as a normal solution arc if there are

$$\Omega = p - \bar{p} + 2N$$

variations $\delta_\omega a \quad \omega = 1,\ldots,\Omega$ which satisfy the conditions

(23a) $\delta_\omega \psi^\alpha(t) = 0$ on[1] a neighborhood of $S^\alpha \quad \alpha = 1,\ldots,m$

(23b) $\delta_\omega \psi^\alpha(t^0) = 0$ if $\mu_\alpha(t^0) \neq \lambda_{p+N+\alpha} \quad \alpha = 1,\ldots,m$

(23c) $\delta_\omega \dot{x}^i = \delta_\omega f^i \quad i = 1,\ldots,N$

$\left(\text{where } S^\alpha = \left\{t \mid \psi^\alpha(t) = 0\right\}\right)$ in addition to providing that the matrix

(23d) $\left(J'_\rho(a_0, \delta_\omega a)\right) \quad \rho = 1,\ldots,p+2N \quad \rho \neq \gamma_k \quad \omega = 1,\ldots,\Omega$

is non-singular

Next, define the class Y of variations

y: $\quad y^{i(t)}, \quad v^k(t) \quad z^\sigma \quad t^0 \leq t \leq t^1$

of a_0 which satisfy

(24a) $\dot{y}^i = \delta_y f^i \quad i = 1,\ldots,N$

(24b) $\delta_y \psi^\alpha(t) \leq 0$ on a neighborhood of $S^\alpha \quad \alpha = 1,\ldots,m$

(24c) $\delta_y \psi^\alpha(t^c) = 0$ if $\mu_\alpha(t^c) \neq \lambda_{p+N+\alpha} \quad c = 0,1$

(24d) $\delta_y \psi^\alpha(\bar{t}) = 0$ if $\mu_\alpha(t)$ is not constant on a neighborhood of $\bar{t} \quad \alpha = 1,\ldots,m$

(24e) $\delta_y \psi^\alpha(t^0) = 0$ if $\psi^\alpha(t^0) = 0 \quad \alpha = 1,\ldots,m$

(1) As defined previously, the term $\delta_\omega \psi^\alpha(t) = \psi^\alpha_{x^i}(t)\delta_\omega x^i(t)$ is the variation in the function ψ^α due the variation $\delta_\omega a$ of a_0 .

(24f) $$J'_\gamma(a_0, y) = 0 \quad \text{if } \lambda_\gamma \neq 0 \quad 1 \leq \gamma \leq p'$$

(24g) $$J'_\gamma(a_0, y) \leq 0 \quad \text{if } \lambda_\gamma = 0 \quad \gamma \neq \gamma_k \quad 1 \leq \gamma \leq p'$$

(24h) $$J'_\rho(a_0, y) = 0 \quad p' < \rho \leq p + 2N$$

where, $\mu_\alpha(t)$, $\lambda_{p+N+\alpha}$, λ_γ are multipliers of Theorem 3.2 of [1].

Finally, we list the functions H and G of Theorem 3.2 of [1]

$$H = p_i(t)f^i - \lambda_0 L_0 - \lambda_\gamma L_\gamma - \mu_\alpha(t)\phi^\alpha$$

$$G = \lambda_0 g_0 + \lambda_\gamma g_\gamma + \lambda_{p+i} X^{i0} - \lambda_{p+N+\alpha} [\psi^\alpha(t^s, X^s)]_{s=0}^{s=1}$$

$$i = 1,\ldots,N \quad \gamma = 1,\ldots,p \quad \alpha = 1,\ldots,m$$

where $p(t)$, λ, $\mu(t)$ are multipliers of the theorem referred to above,

The second order theorem to be proven for the problem of section 3 is then:

Theorem 4.1 . Let a_0 be a normal solution to the problem of section 3. Then the multipliers $\lambda_\rho, \mu_\alpha(t)$, $p_i(t), K^\alpha$ $\rho = 0,1,\ldots,p+N+m$

$\alpha = 1,\ldots,m$, $i = 1,\ldots,N$ of Theorem 3.2 of [1] satisfy: (i) $\lambda_0 \neq 0$ and (ii) if $\lambda_0 = 1$, then the multipliers $\lambda_0, \lambda_1, \ldots, \lambda_{p+N}, K^\alpha$ are unique while the multipliers $\mu_\alpha(t), p_i(t), \lambda_{p+N+\alpha}$ $(\alpha = 1,\ldots,m)$ are unique up to the respective added terms c^α, $c^\alpha \psi^\alpha_{x^i}(t)$, c^α for arbitrary c^α. Furthermore, with H, G as the functions listed above, then for each variation y in the class Y we have that

$$(25)\qquad 0 \le [(p_i(t^c)X^{ic}_{b^\sigma b^\tau})\Big|_{c=0}^{c=1} + G_{b^\sigma b^\tau} + K^\alpha \psi^\alpha_{x^i x^j}(t^0)X^{i0}_{b^\sigma}X^{j0}_{b^\tau}]z^\sigma z^\tau$$

$$- \int_{t^0}^{t^1} [H_{x^i x^j} y^i y^j + 2H_{x^i u^h} y^i v^h + H_{u^h u^k} v^h v^k]\, dt$$

$$i,j = 1,\dots,N \qquad \sigma,\tau = 1,\dots,r \qquad h,k = 1,\dots,K$$

5. <u>Transformation of the Problem of Section 3</u>

Now according to [1], the problem of section 3 may be transformed into a canonical problem by introducing the variables

$$(26)\qquad x^{N+\alpha}, \quad u^{K+\alpha}, \quad (\alpha=1,\dots,m)\ ; \quad b^{r+s}, \quad (s=1,\dots,2m)$$

and the conditions

$$(27)\qquad \dot{x}^{N+\alpha} = f^{N+\alpha} = u^{K+\alpha}, \quad x^{N+\alpha} < 0 \quad \alpha = 1,\dots,m$$

and by defining the functions (without summing on α)

$$\bar{\psi}^\alpha(t,x) = \psi^\alpha(t,x) + x^{N+\alpha}\psi^\alpha(t)$$

$$\bar{\phi}^\alpha(t,x,u) = \phi^\alpha(t,x,u) + u^{K+\alpha}\psi^\alpha(t) + x^{N+\alpha}\phi^\alpha(t) \quad \alpha = 1,\dots,m$$

(1)
where $\psi^\alpha(t)$ and $\phi^\alpha(t)$ refer to values along the solution arc a_0. In addition, let $\bar{R}$, $\bar{B}$ be defined as the respective Cartesian Products $R \times \mathcal{X}^{N+1} \times \dots \times \mathcal{X}^{N+m} \times \mathcal{U}^{K+1} \times \dots \times \mathcal{U}^{K+m}$ and $B \times B^1 \times \dots \times B^{2m}$ where for example $\mathcal{X}^{N+\alpha}$ is the space of the variable $x^{N+\alpha}$.

We are now concerned with arcs in $\bar{R}$. Such an arc will be designated by $\bar{a}$ where this symbol denotes

(1) It will frequently be necessary to distinguish between quantities in the original and transformed problems. We will generally use the "barred" quantities as e.g. $\bar{\psi}^\alpha$ for the latter and "unbarred" quantities as ψ^α for the former. If the "unbarred" is used in both versions, then the same quantity is indicated in both versions.

$\bar{a}$: $\quad \bar{x}^j(t) \quad \bar{u}^k(t) \quad \bar{b}^\sigma \quad t^0 \le t \le t^1$

$$j = 1,\ldots N{+}m, \quad k = 1,\ldots,K{+}m, \quad \sigma = 1,\ldots,r{+}2m.$$

The transformed problem is then to minimize the integral

$$(30) \qquad I_0(\bar{a}) = g_0(\bar{b}) + \int_{t^0}^{t^1} L_0(t,\bar{x}(t),\bar{u}(t))dt$$

on the class $\bar{C}$ of all arcs $\bar{a}$ which have (i) $(t,\bar{x}(t),\bar{u}(t))$ in $\bar{R}$, (ii) $\bar{b}$ in $\bar{B}$, (iii) $\bar{u}(t)$ piecewise continuous and in addition satisfy the conditions

$$\dot{\bar{x}}^j = f^j \qquad j = 1,\ldots,N{+}m$$

$$(31) \qquad \bar{\psi}^\alpha \le 0 \qquad \alpha = 1,\ldots,m$$

$$\bar{x}^{N+\alpha} < 0 \qquad \alpha = 1,\ldots,m$$

$$I_\gamma \le 0 \quad (\gamma=1,\ldots,p') \qquad I_\gamma = 0 \quad (\gamma=p'+1,\ldots,p)$$

$$\bar{x}^j(t^s) = X^{js} \qquad j = 1,\ldots,N{+}m; \quad s = 0,1$$

where the functions $X^{is}(b)$ $(s = 0,1 \quad i = 1,\ldots,N)$ have the meaning already specified and

$$(32) \qquad X^{N+\alpha,0}(b) \equiv b^{r+\alpha} \qquad X^{N+\alpha,1}(b) = b^{r+m+\alpha} \qquad \alpha = 1,\ldots,m$$

Define the arc

$$(33) \qquad \bar{a}_0: \quad \bar{x}_0{}^j(t) \quad \bar{u}_0{}^k(t) \quad \bar{b}_0{}^\sigma \quad t^0 \le t \le t^1$$

with $\quad \bar{x}_0^i = x_0^i \quad (i = 1,..,N); \quad \bar{x}_0^{N+\alpha} = -\theta \quad (\alpha=1,\ldots,m)$

$$\bar{u}_0^h = u_0^h \quad (h = 1,\ldots,K); \quad \bar{u}_0^{K+\alpha} = 0 \quad (\alpha=1,\ldots,m)$$

$$\bar{b}_0^\sigma = b_0^\sigma \quad (\sigma = 1,\ldots,r); \quad \bar{b}_0^{r+\alpha} = \bar{b}_0^{r+m+\alpha} = -\theta \quad (\alpha = 1,\ldots,m)$$

where θ is a constant satisfying $0 < \theta < 1$.

By the reasoning of [1]: (i) the arc $\bar{a}_0$ is a solution to the transformed problem if the arc a_0 of (17) is a solution to the original problem (of section 3) and (ii) the assumption involving (5) holds true for the transformed problem. According to the continuity assumptions on the functions ψ^α, f^i, L_γ, g_γ, X^{is}, $u_0(t)$ and the definitions (27), (32), and (33), we see that the continuity assumptions required for the canonical problem and listed below (3) are satisifed also by this transformed problem. Thus according to Theorem 6.1 and section 7 of [1] there are the multipliers $\bar{\lambda}_\rho$, $\bar{p}_j(t)$, $\bar{\mu}_\alpha(t)$, $\bar{K}^\alpha$, $\rho = 0, 1, \ldots p+N+m$, $j = 1, \ldots, N+m$, $\alpha = 1, \ldots, m$ and functions

(34a) $$\bar{H} = \bar{p}_i(t)f^i - \bar{\lambda}_0 L_0 - \bar{\lambda}_\gamma L_\gamma - \bar{\mu}_\alpha(t)\phi^\alpha + \bar{p}_{N+\alpha}(t)\, u^{K+\alpha} - \bar{\mu}_\alpha(t)[x^{N+\alpha}\phi^\alpha(t) + u^{K+\alpha}\psi^\alpha(t)]$$

$$\bar{G} = \bar{\lambda}_0 g_0 + \bar{\lambda}_\gamma g_\gamma + \bar{\lambda}_{p+i} X^{i0} + \bar{\lambda}_{p+N+\alpha} \bar{b}^{r+\alpha}$$

$i = 1, \ldots, N;$ $\gamma = 1, \ldots, p;$ $\alpha = 1, \ldots, m$

where α is not summed within the brackets and the terms of (34a) are from Theorem 6.1 of [1]. The relationship between the terms of Theorems 6.1 and 3.2 of [1] as explained in sections 7 and 3 of [1] are as follows:

(34b) $$\lambda_\rho = \bar{\lambda}_\rho \qquad \rho = 0, 1, \ldots, p+N$$

$$K^\alpha = \bar{K}^\alpha \qquad \alpha = 1, \ldots, m$$

(34b) $$\mu_\alpha(t) = \bar{\mu}_\alpha(t) + \lambda_{p+N+\alpha} \qquad \alpha = 1,..,m$$

$$p_i(t) = \bar{p}_i(t) + \lambda_{p+N+\alpha}\psi^\alpha_{x^i}(t) \qquad i = 1,\ldots,N \qquad \alpha = 1,\ldots,m$$

$$H = \bar{H} - \bar{p}_{N+\alpha}(t)u^{K+\alpha} + \bar{\mu}_\alpha(t)\,[x^{N+\alpha}\phi^\alpha(t) + u^{K+\alpha}\psi^\alpha(t)] + \lambda_{p+N+\alpha}\psi^\alpha_{x^i}(t)f^i -$$
$$\lambda_{p+N+\alpha}[\psi^\alpha_t + \psi^\alpha_{x^i}f^i] \qquad i = 1,\ldots N \qquad \alpha = 1,\ldots,m$$

$$G = \bar{G} - \bar{\lambda}_{p+N+\alpha}\; b^{r+\alpha} - \lambda_{p+N+\alpha}[\psi^\alpha(t^c X^c)]_{c=0}^{c=1} \qquad \alpha = 1,\ldots,m$$

where: (i) $\lambda_{p+N+\alpha}$ $(\alpha = 1,\ldots,m)$ are m arbitrary constants[(1)] and (ii) α is not summed within the brackets in the above expression and (iii) the terms $\lambda_{p+N+\alpha}\;\psi^\alpha_{x^i}(t)$, added to the right hand side in the fourth relation above, are to be considered as part of the multipliers of the H function (see section 3 of [1]).

In order now to use Theorem 2.1 for the transformed problem we must show that the transformed problem is normal. This is done in the following lemma.

Lemma 5.1. Normality of the original problem (of section 3) implies normality of the transformed problem.

Using the variations $\delta_\omega\alpha$ $(\omega=1,\ldots,\Omega)$ defined in (23) we define variations for the transformed problem as follows:

(35) $$\bar{\delta}_\omega \bar{a}: \quad \bar{\delta}_\omega\bar{x}^j(t) \quad \bar{\delta}_\omega\bar{u}^k(t) \quad \bar{\delta}_\omega\bar{b}^\sigma \qquad t^0 \le t \le t^1$$

$$j = 1,\ldots,N+m \qquad k = 1,\ldots,K+m \qquad \sigma = 1,\ldots,r+2m$$

(1) Notice that in general $\bar{\lambda}_{p+N+\alpha} \neq \lambda_{p+N+\alpha}$ $(\alpha = 1,\ldots,m)$

with:

$$(36)\qquad \bar{\delta}_\omega \bar{x}^i = \delta_\omega x^i \quad i = 1,\dots,N \qquad \bar{\delta}_\omega \bar{x}^{N+\alpha} = 0 \quad \alpha = 1,\dots,m$$

$$\bar{\delta}_\omega \bar{u}^h = \delta_\omega u^h \quad h = 1,\dots,K \qquad \bar{\delta}_\omega \bar{u}^{K+\alpha} = 0 \quad \alpha = 1,\dots,m$$

$$\bar{\delta}_\omega \bar{b}^\sigma = \delta_\omega b^\sigma \quad \sigma = 1,\dots,r \qquad \bar{\delta}_\omega \bar{b}^{r+s} = 0 \quad s = 1,\dots,2m$$

where the unbarred quantities are from (23), and the unlisted argument in the variations for x, $\bar{x}$, u, $\bar{u}$, is t, $(t^0 \le t \le t^1)$. With this construction and (23) we see that

$$(37a)\qquad \bar{\delta}_\omega \bar{\psi}^\alpha(t) = \delta_\omega \psi^\alpha(t) \qquad t^0 \le t \le t^1 \qquad \alpha = 1,\dots,m$$

$$(37b)\qquad \bar{\delta}_\omega \dot{\bar{x}}^i = \delta_\omega \dot{x}^i = \delta_\omega f^i = \bar{\delta}_\omega f^i \qquad i = 1,\dots,N$$

$$(37c)\qquad \bar{\delta}_\omega \dot{\bar{x}}^{N+\alpha} = \bar{\delta}_\omega f^{N+\alpha} = 0 \qquad \alpha = 1,\dots,m \qquad \omega = 1,\dots,\Omega$$

Then by the last two relations, these variations satisfy condition (9a) for the transformed problem. Furthermore according to the construction of the arc $\bar{a}_0$ and the definition of the functions $\bar{\psi}^\alpha$ which imply that[(1)]

$$(37d)\qquad \bar{\psi}^\alpha(t) = 0 \quad \text{iff} \quad \psi^\alpha(t) = 0 \qquad \alpha = 1,\dots,m$$

then the sets S^α for $\bar{a}_0$ and for a_0 are the same. Thus the properties of the variations $\delta_\omega a$ imply that the variations $\bar{\delta}_\omega \bar{a}$ also satisfy condition (9b) and are then admissible.

Next, according to (34b) we see that if

$$(37e)\qquad \mu_\alpha(t^0) \ne \lambda_{p+N+\alpha} \qquad \alpha = 1,\dots,m$$

(1) The terms $\bar{\psi}^\alpha(t)$, $\psi^\alpha(t)$ are evaluated along $\bar{a}_0$, a_0 respectively

then the associated multipliers of Theorem 6.1 of [1] satisfy

(37f) $$\bar{\mu}_\alpha(t^0) \neq 0 \qquad \alpha = 1,\dots,m$$

Then by using the above determined facts about the sets S^α, we see that properties (23a) and (23b) for the variations $\delta_\omega a$ imply properties (12a) and (12b) for the variations $\bar{\delta}_\omega \bar{a}$. In addition, according to definition, the functional $I_\gamma(\bar{a})$, does not depend upon the variables $x^{N+\alpha}$, $u^{K+\alpha}$, b^{r+s}. Thus with the above definitions of the arc $\bar{a}_0$ and of the variations $\bar{\delta}_\omega \bar{a}$, we see that

(38a) $$I_\gamma(\bar{a}_0) = I_\gamma(a_0) \qquad \gamma = 0,1,\dots,p$$

(38b) $$I_\gamma'(\bar{a}_0,\bar{\delta}_\omega \bar{a}) = I_\gamma'(a_0,\delta_\omega a) \qquad \omega = 1,\dots\Omega$$

and furthermore, since the functions X^{is} $i = 1,\dots,N$, $s = 0,1$ are independent of $b^{r+1},\dots,b^{r+2m}$, also then

(38c) $$\bar{\delta}_\omega \bar{x}^i(t^1) - \bar{\delta}_\omega X^{i1} = \delta_\omega x^i(t^1) - \delta_\omega X^{i1}$$

(38d) $$\bar{\delta}_\omega X^{i0} - \bar{\delta}_\omega \bar{x}^i(t^0) = \delta_\omega X^{i0} - \delta_\omega x^i(t^0) \qquad i = 1,\dots,N;\ \omega = 1,\dots,\Omega$$

Defining now the $2m$ additional variations $\bar{\delta}_{\Omega+1}\bar{a},\dots,\bar{\delta}_{\Omega+2m}\bar{a}$ as:

(39) $$\bar{\delta}_{\Omega+s}\bar{a}: \qquad \bar{\delta}_{\Omega+s}\bar{x}^j \equiv 0 \ , \qquad \bar{\delta}_{\Omega+s}\bar{u}^k \equiv 0 \ , \qquad \bar{\delta}_{\Omega+s}\bar{b}^\sigma = 0 \ , \qquad \bar{\delta}_{\Omega+s}\bar{b}^{r+s} = 1$$

$$s = 1,\dots,2m \qquad j = 1,\dots,N+m \qquad k = 1,\dots K+m \qquad \sigma = 1,\dots,r+2m \qquad \sigma \neq r+s$$

we see that these variations also satisfy the conditions (9) and (12). Furthermore, according to (38), together with the definition of the functional variations J'_ρ in (20), the property (23d) of the variations $\delta_\omega a$, and the construction (39), then the matrix

$$(40)\qquad \begin{pmatrix} I'_\gamma\,(\bar{a}_0\,,\,\bar{\delta}_\omega\bar{a}) \\ \bar{\delta}_\omega\bar{x}^j(t^1) - \bar{\delta}_\omega X^{j1} \\ \bar{\delta}_\omega X^{j0} - \bar{\delta}_\omega\bar{x}^j(t^0) \end{pmatrix} \qquad \begin{array}{l} \gamma = 1,\ldots,p \quad . \quad \gamma \neq \gamma_k \\ j = 1,\ldots,N+m \quad \omega = 1,\ldots,\Omega + 2m \end{array}$$

is non singular (where γ_k are the indices of (21)).

Now analagous to (6) the functionals which we shall use to determine normality for the transformed problem are

$$\bar{J}_\gamma(\bar{a}) = q_{\gamma_j}(t^0)[\bar{x}^j(t^0) - X^{j0}(\bar{b})] + I_\gamma(\bar{a}) \qquad \gamma = 1,\ldots,p.$$

$$(41)\qquad \bar{J}_{p+i}(a) = z_{ij}(t^0)[X^{j0}(\bar{b}) - \bar{x}^j(t^0)] + \bar{x}^i(t^1) - X^{i1}(\bar{b})$$

$$\bar{J}_{p+N+i}(a) = X^{i0}(\bar{b}) - \bar{x}^i(t^0) \qquad j,i = 1,\ldots,N+m$$

where $q_{\gamma_j}(t)$, $z_{ij}(t)$ are as described there but for the transformed problem.

Forming the variations of these functionals for the variations of (36) and (39) and for the indicated range of γ , we get the matrices

$$(42)\qquad \begin{pmatrix} \bar{J}_\gamma{}'(\bar{a}_0,\bar{\delta}_\omega\bar{a}) \\ \\ \bar{J}'_{p+i}(\bar{a}_0,\bar{\delta}_\omega\bar{a}) \\ \\ \bar{J}'_{p+N+i}(\bar{a}_0,\bar{\delta}_\omega\bar{a}) \end{pmatrix} = \begin{pmatrix} q_{\gamma_j}(t^0)[\bar{\delta}_\omega\bar{x}^j(t^0) - \bar{\delta}_\omega X^{j0}] + I_\gamma{}'(\bar{a}_0,\bar{\delta}_\omega\bar{a}) \\ \\ z_{ij}(t^0)[\bar{\delta}_\omega X^{j0} - \bar{\delta}_\omega\bar{x}^j(t^0)] + \bar{\delta}_\omega\bar{x}^i(t^1) - \bar{\delta}_\omega X^{i1} \\ \\ \bar{\delta}_\omega X^{i0} - \bar{\delta}_\omega\bar{x}^i(t^0) \end{pmatrix}$$

$\gamma = 1,\ldots,p \qquad \gamma \neq \gamma_k \qquad j,i = 1,\ldots,N+m \qquad \omega = 1,\ldots,\Omega+2m$

where γ_k are the indices of (21)

Now for each row, the last terms in the right matrix of (42) constitute the terms of the matrix of (40) and the other terms in each row of (42) are linear combinations of the last rows of the matrix of (40). Thus since the rank of a matrix is unaltered by adding linear combinations of its rows to other rows, the non-singularity of the matrix of (40) implies the non-singularity of the matrix of (42).

Finally, according to (38a) the indices γ_k determined as in (11) but for the transformed problem are the same as the indices γ_k from (21) and the transformed problem is then normal, proving the lemma.

Next, corresponding to the class of variations defined below (12) define the class $\bar{Y}$ of variations for the transformed problem as those admissible variations.

$$\bar{y}: \qquad \bar{y}^j(t) \qquad \bar{v}^k(t) \qquad \bar{z}^\sigma \qquad t \leq t \leq t^1$$

of $\bar{a}_0$ which satisfy

(43a) $\bar{\delta}_{\bar{y}}\bar{\psi}^{\alpha}(t^{c}) = 0$ if $\bar{\mu}_{\alpha}(t^{c}) \neq 0 \qquad c = 0,1$

(43b) $\bar{\delta}_{\bar{y}}\bar{\psi}^{\alpha}(\bar{t}) = 0$ if $\bar{\mu}_{\alpha}(t)$ is not constant on a neighborhood of $\bar{t}$

(43c) $\bar{\delta}_{\bar{y}}\bar{\psi}^{\alpha}(t^{0}) = 0$ if $\bar{\psi}^{\alpha}(t^{0}) = 0 \qquad \alpha = 1,\ldots,m$

(43d) $\bar{J}'_{\gamma}(\bar{a}_0,\bar{y}) = 0$ if $\bar{\lambda}_{\gamma} \neq 0 \qquad \gamma = 1,\ldots,p'$

(43e) $\bar{J}'_{\gamma}(\bar{a}_o,\bar{y}) \leq 0$ if $\bar{\lambda}_{\gamma} = 0 \qquad \gamma \neq \gamma_k \qquad 1 \leq \gamma \leq p'$

(43f) $\bar{J}'_{\rho}(\bar{a}_0,\bar{y}) = 0 \qquad p' < \rho \leq p + 2(N+m)$

where $\bar{J}'_{\rho}$ are the variations of the functionals (41) for the transformed problem and γ_k are the indices of (21). With this definition we now prove :

Lemma 5.2 for each variation y in the class Y there is a **variation** $\bar{y}$ in the class $\bar{Y}$.

Proof: The proof of this lemma is directly analagous to part of the proof of Lemma 5.1. Given the variation y in Y

(44) y: $\qquad y^{i}(t) \qquad v^{k}(t) \qquad z^{\sigma} \qquad t^{o} \leq t \leq t^{1}$

define the variation $\bar{y}$ in the class $\bar{Y}$ as follows,

(45) $\bar{y}^{i} = y^{i} \qquad i = 1,\ldots,N \qquad \bar{y}^{N+\alpha} = 0 \qquad \alpha = 1,\ldots,m$

$\bar{v}^{k} = v^{k} \qquad k = 1,\ldots,K \qquad \bar{v}^{N+\alpha} = 0 \qquad \alpha = 1,\ldots,m$

$\bar{z}^{\sigma} = z^{\sigma} \qquad \sigma = 1,\ldots,r \qquad \bar{z}^{r+s} = 0 \qquad s = 1,\ldots,2m$

By reasoning exactly as in Lemma 5.1, we see that this variation $\overline{\mathcal{Y}}$ is admissible and also satisfies conditions (43a) through (43c). By construction and the properties of $\mathcal{Y}$, also (43f) is satisfied by $\overline{\mathcal{Y}}$. Then by (38a) together with reasoning as used in obtaining (38b) and the relations (34b) between the multipliers we see that also (43d) and (43e) are satisfied by $\overline{\mathcal{Y}}$. Then $\overline{\mathcal{Y}}$ is in the class $\overline{Y}$ and the lemma is proven.

We are now in position to prove Theorem 4.1 from Theorem 2.1

6. Proof of Theorem 4.1

Lemma 4.1: Theorem 2.1 implies Theorem 4.1

According to Theorem 2.1 applied to the transformed problem, the multipliers

$$\overline{\lambda}_\rho,\ \overline{\mu}_\alpha(t), \overline{p}_j(t), \overline{K}^\alpha,\ \rho = 0,1,\ldots,p + N + m,\quad \alpha = 1,\ldots,m,\ j = 1,\ldots,N + m$$

of (34a) satisfy

$$\overline{\lambda}_0 \neq 0 \tag{46}$$

and if these multipliers are selected with $\overline{\lambda}_0 = 1$, then they are unique.

Now according to the relationship (34b) between the multipliers of Theorems 3.2 and 6.1 of [1], we see that the first two statements of Theorem 4.1 are proven.

In order to establish the inequality (25), let $\mathcal{Y}$ be a variation in the class Y and construct the variation $\overline{\mathcal{Y}}$ from Y as in (45). Then by (14) applied to the transformed problem we have

$$\begin{aligned} 0 \le {} & [(\overline{p}_j(t^c)X^{jc}_{b^\sigma b^\tau})^{c=1}_{c=o} + \overline{G}_{b^\sigma b^\tau} + \overline{K}^\alpha\overline{\psi}^\alpha_{x^i x^j}(t^0)X^{i0}_{b^\sigma}X^{j0}_{b^\tau}]\overline{z}^\sigma\overline{z}^\tau - \\ & \int_{t^0}^{t^1} \overline{H}_{x^i x^j}\overline{y}^i\overline{y}^j + 2\overline{H}_{x^i u^h}\overline{y}^i\overline{v}^h + \overline{H}_{u^h u^k}\overline{v}^h\overline{v}^k]dt \end{aligned} \tag{47}$$

$$\sigma,\tau = 1,\ldots,r \qquad i,j = 1,\ldots,N \qquad h,k = 1,\ldots,k$$

where in using the indicated index ranges in (47) we have recognized that: i) the functions X^{jc} $(j = 1,\ldots,N)$ do not depend on b^{r+s} $(s = 1,\ldots,2m)$, ii) the functions $X^{N+\alpha,c}$ $(\alpha = 1,\ldots,m;\ c = 0,1)$ are linear and so have zero second partial derivatives; iii) the functions $\overline{H}$ of (34a) and $\overline{\psi}^{\alpha}$ of (28) satisfy

$$(48)\quad \overline{H}_{x^j x^{N+\alpha}} = 0 \qquad \overline{\psi}^{\alpha}_{x^j x^{N+\alpha}} = 0 \qquad \alpha = 1,\ldots,m \qquad j = 1,\ldots,N+m$$

$$\overline{H}_{x^{N+\alpha} u^h} = 0 \qquad h = 1,\ldots,k+m$$

$$\overline{H}_{x^j u^{k+\alpha}} = 0 \qquad j = 1,\ldots,N+m$$

$$\overline{H}_{u^h u^{k+\alpha}} = 0 \qquad h = 1,\ldots,k+m \qquad \alpha = 1,\ldots,m$$

Furthermore by (34b), (see the remarks below (34b)) and (28) we see that

$$(49)\quad \overline{H}_{x^j x^k} = H_{x^j x^k} + \lambda_{p+N+\alpha}[\psi^{\alpha}_{tx^j x^k} + \psi^{\alpha}_{x^i x^j x^k} f^i + \psi^{\alpha}_{x^i x^j} f^i_{x^k} + \psi^{\alpha}_{x^i x^k} f^i_{x^j}]$$

$$\overline{H}_{x^j u^h} = H_{x^j u^h} + \lambda_{p+N+\alpha}[\psi^{\alpha}_{x^i x^j} f^i_{u^h}]$$

$$\overline{H}_{u^h u^r} = H_{u^h u^r} \qquad h,r = 1,\ldots,k$$

$$\overline{\psi}^{\alpha}_{x^j x^k} = \psi^{\alpha}_{x^j x^k} \qquad i,j,k = 1,\ldots,N$$

where as usual all derivatives are formed along the solution arc which is $\overline{a}_o$ in the present case.

Using the relations (34b) again and (48) together with (49) and the construction of $\mathcal{Y}$, we may write (47) as

$$(50)\quad 0 \le \Big[(p^i(t^c) - \lambda_{p+N+\alpha}\psi^\alpha_{x^i}(t^c))X^{ic}_{b^\sigma b^\tau} + \lambda_{p+N+\alpha}(\psi^\alpha_{x^i}(t^c)X^{ic}_{b^\sigma b^\tau} + \psi^\alpha_{x^i x^j}(t^c)X^{ic}_{b^\sigma}X^{jc}_{b^\tau}) \Big]_{c=0}^{c=1} z^\sigma z^\tau$$

$$+ [K^\alpha \psi^\alpha_{x^i x^j}(t^0)X^{i0}_{b^\sigma}X^{j0}_{b^\tau} + G_{b^\sigma b^\tau}]z^\sigma z^\tau - \int_{t^0}^{t^1}[H_{x^i x^j}y^i y^j + 2H_{x^i u^h}y^i v^h + H_{u^h u^k}v^h v^k]dt$$

$$- \lambda_{p+N+\alpha}\int_{t^0}^{t^1}[(\psi^\alpha_{tx^j x^i} + \psi^\alpha_{x^s x^j x^i}f^s + \psi^\alpha_{x^s x^j}f^s_{x^i} + \psi^\alpha_{x^s x^i}f^s_{x^j})y^j y^i + 2\psi^\alpha_{x^s x^j}f^s_{u^h}y^j v^h]dt$$

$i,j,s = 1,\dots,N; \quad \alpha = 1,\dots,m; \quad k,h = 1,\dots,k; \quad \sigma,\tau = 1,\dots,r$

Now, recognizing that

$$(51)\qquad \frac{d}{dt}(\psi^\alpha_{x^j x^i}) = \psi^\alpha_{tx^j x^i} + \psi^\alpha_{x^s x^j x^i}f^s$$

and by (24a) together with the construction of $\overline{\mathcal{Y}}$, that

$$(52)\quad \psi^\alpha_{x^s x^j}(f^s_{x^i}y^i + f^s_{u^h}v^h) = \psi^\alpha_{x^s x^j}\dot{y}^s \qquad s,j,i = 1,\dots,N \qquad h = 1,\dots,k$$

and similarly for the remaining terms, we can write the last integral in (50) as

$$(53)\quad - \lambda_{p+N+\alpha}\int_{t^0}^{t^1}[\frac{d}{dt}(\psi^\alpha_{x^j x^i})y^j y^i + \psi^\alpha_{x^j x^i}(\dot{y}^i y^j + y^i\dot{y}^j)]dt$$

$$= - \lambda_{p+N+\alpha}\int_{t^0}^{t^1}\frac{d}{dt}(\psi^\alpha_{x^j x^i}y^j y^i)dt = - \lambda_{p+N+\alpha}[\psi^\alpha_{x^j x^i}(t^c)X^{ic}_{b^\sigma}X^{jc}_{b^\tau}]_{c=0}^{c=1} z^\sigma z^\tau$$

where in the last equality of (53) we have used the property (24h) of the variation $\mathcal{Y}$.

Then by putting together (53) and (50) we obtain (25) thus proving Theorem 4.1.

7. Further Generalizations

The remainder of this paper will consist of three additional generalizations. Each generalization will be handled in a manner analogous to the previous one. Thus we will in each case i) state the generalized problem considered, ii) define normality for that problem, iii) define a particular class of variations of the solutions arc, iv) state the theorem to be proven and v) prove that theorem by referring back to the theorem of the previous generalization considered. In the interest of conciseness, when the steps of the proof are directly analogous to those of previous sections, we shall omit some of the detail of those steps, instead making reference to the corresponding item in the previous sections.

8. A Problem With Constraints Involving u and x

As a next step in our generalization process, consider the problem of section 3 with constraints involving the control variables u and state variables x as

(54) $$\theta(t,x,u) = 0$$

Thus with all quantities having the meaning of section 3, we desire to minimize

(55) $$I_0(a) = g_0(b) + \int_{t^0}^{t^1} L_0(t,x,u)dt$$

among arcs as described in section 3 which satisfy the conditions

(56a) $$\dot{x}^i = f^i \qquad i = 1, \ldots, N$$

(56b) $$\psi^\alpha(t,x) \le 0 \qquad \alpha = 1, \ldots, m$$

(56c) $$\theta^{\eta}(t,x,u) = 0 \qquad \eta = 1, \ldots, L$$

(56d) $$I_{\gamma}(a) \leq 0 \qquad \gamma = 1,\ldots,p' \ ; \qquad I_{\gamma}(a) = 0 \qquad \gamma = p' + 1,\ldots,p$$

(56e) $$x^{i}(t^{s}) = X^{is}(b) \quad s = 0, 1 \qquad i = 1, \ldots, N$$

where as in section 3

(57) $$I_{\gamma}(a) = g_{\gamma}(b) + \int_{t^0}^{t^1} L_{\gamma}(t,x,u)dt \qquad \gamma = 1, \ldots, p$$

Now given that the arc

(58a) $$a_0: \qquad x_0(t) \qquad u_0(t), \qquad b_0 \qquad t^0 \leq t \leq t^1$$

is a solution to this problem, we assume the analogous continuity conditions listed below (17a) also for the present problem in addition to assuming that the functions $\theta^{\eta}(\eta = 1, \ldots, L)$ are of class C^2 on R. The sets R, R_0, R_1, are defined here in an analogous manner to that of section 3. Corresponding to the assumption involving the matrix of (17c), we assume that the matrix

(58b) $$\begin{bmatrix} \theta^{\eta}_{u^k} & 0 \\ \phi^{\alpha}_{u^k} & \delta_{\alpha\beta}\psi\beta \end{bmatrix} \qquad \alpha,\beta = 1, \ldots, m \quad \eta = 1, \ldots, L \ ; \ k = 1, \ldots, K$$

(where $\phi^{\alpha} = \psi^{\alpha}_{t} + \psi^{\alpha}_{x^i} f^{i}$) has rank m on $\overline{R}_1$, the closure of R_1 in R.

We will have need to refer to Theorem 6.1 of [2] for this problem. For later results we now add one statement henceforth referred to as the "extra statement" which is the analogue of (22) of [1] and is as follows:

With λ_ρ, K^α, $p_i(t)$ as multipliers of Theorem 6.1[1] of [2], then for the case when

$$\lambda_\gamma = 0 \text{ if } \gamma \notin L (0 \le \gamma \le p); \; K^\alpha = 0, \quad \lambda_{p+N+\alpha} = \mu_\alpha(\bar{t}) = C^\alpha \quad \alpha = 1,\ldots,m$$

and

$$p_j(\bar{t}) = C^\alpha \psi^\alpha_{x^j}(\bar{t}) \quad j \notin K, \qquad p_i(t_i) = C^\alpha \psi^\alpha_{x^i}(t_i) \quad i \in K.$$

for any points $\bar{t}$, t_i in $[t^0, t^1]$ and any constants C^α $(\alpha = 1,\ldots,m)$ and where K, L are the respective sets of indices $1 \le i \le N$ $\quad 0 \le \gamma \le p$ for which $f^i = L_\gamma \equiv 0$,[2] then we must have

$$p_i(t) = C^\alpha \psi^\alpha_{x^i}(t) \qquad t^0 \le t \le t^1$$

The proof of this statement follows along the same lines as the proofs of (34) to (36) and (11) of [2].

Normality for the present problem is defined in an analagous fashion to that used in section 4 as long as we take account of the additional constraints (54).

(1) We are omitting the variable time aspect of these problems and so the problem of section 2 of [2] and Theorem 3.1 of [2] are identical with the problem of section 2 of [1] and Theorem 3.2 of [1]. Also we do not include in these or later references to theorems of [2] any items involved with the variable time aspects of these problems. Thus e.g., we do not include the λ multipliers of Theorem 6.1 of [2] which are associated with the term T^0 and consider instead that only $p+N+m$ of the λ multipliers are present. There is no difficulty in doing this since the proofs of those parts of the Theorems of [2] not dealing with variable time aspects, hold also when the variable time problem is replaced by the corresponding fixed time problem in each case.

(2) The notation $f^i \equiv 0$ means that f^i is identically zero on its region of definition and a similar remark holds for $L_\gamma \equiv 0$.

Let δa

(59) δa: $\quad \delta x(t) \quad \delta u(t) \quad \delta b \quad t^0 \le t \le t^1$

denote a variation of the arc a_0 of (58a) and define for this problem exactly the same functions and their variations as were defined in (19) and (20) and also let γ_k $k = 1,\dots,\bar{p}$ be defined in a manner analagous to that used in (21). Then with $\mu_\alpha(t)$, $\lambda_{p+N+\alpha}$ as the terms of Theorem 6.1 of [2] we call the arc a_0 a normal solution to our problem if there are

(60) $$\Omega = p - \bar{p} + 2N$$

variations $\delta_\omega a$ $(\omega = 1,\dots,\Omega)$ which satisfy the conditions

(61a) $$\delta_\omega \dot{x}^i = \delta_\omega f^i \qquad i = 1,\dots,N$$

(61b) $$\delta_\omega \psi^\alpha(t^0) = 0 \quad \text{if} \quad \mu_\alpha(t^0) \ne \lambda_{p+N+\alpha}$$

(61c) $$\delta_\omega \psi^\alpha(t) = 0 \quad \text{on a neighborhood of } S^\alpha \qquad \alpha = 1,\dots,m$$

(61d) $$\delta_\omega \theta^\eta(t) = 0 \qquad t^0 \le t \le t^1 \qquad \eta = 1,\dots,L$$

(where as before $S^\alpha = \{t \,|\, \psi^\alpha(t) = 0\}$) in addition to providing that the matrix

(61e) $$(J'_\rho(a_0, \delta_\omega a)) \qquad \rho = 1,\dots,p + 2N \qquad \rho \ne \gamma_k \qquad \omega = 1,\dots,\Omega$$

is non-singular.

Next define the class Y of variations

y: $\quad y^i(t) \quad v^k(t) \quad z^\sigma \quad t^0 \le t \le t^1$

of a_0 which satisfy

(62a) $\dot{y}^i = \delta_y f^i \qquad i = 1,\dots,N$

(62b) $\delta_y \psi^\alpha(t) \le 0$ on a neighborhood of S^α

(62c) $\delta_y \psi^\alpha(t^c) = 0$ if $\mu_\alpha(t^c) \neq \lambda_{p+N+\alpha} \qquad c = 0,1$

(62d) $\delta_y \psi^\alpha(\bar{t}) = 0$ if $\mu_\alpha(t)$ is not constant on a neighborhood of $\bar{t}$

(62e) $\delta_y \psi^\alpha(t^0) = 0$ if $\psi^\alpha(t^0) = 0 \qquad \alpha = 1,\dots,m$

(62f) $\delta_y \theta^\eta(t) = 0 \qquad t^0 \le t \le t^1 \qquad \eta = 1,\dots,L$

(62g) $J'_\gamma(a_0, y) = 0$ if $\lambda_\gamma \neq 0 \qquad 1 \le \gamma \le p'$

(62h) $J'_\gamma(a_0, y) \le 0$ if $\lambda_\gamma = 0 \qquad \gamma \neq \gamma_k \qquad 1 \le \gamma \le p'$

(62i) $J'_\rho(a_0, y) = 0 \qquad p' < \rho \le p + 2N$

where $\mu_\alpha(t)$, $\lambda_{p+N+\alpha}$ are multipliers of Theorem 6.1 of [2].

Lastly, we list the functions H and G of Theorem 6.1 of [2]

$$H = p_i(t) f^i - \lambda_0 L_0 - \lambda_\gamma L_\gamma - \mu_\alpha(t)\phi^\alpha - h_\eta(t)\theta^\eta$$

$$G = \lambda_0 g_0 + \lambda_\gamma g_\gamma + \lambda_{p+i} X^{i0} - \lambda_{p+N+\alpha}\left[\psi^\alpha(t^s, X^s)\right]_{s=0}^{s=1} \qquad \begin{matrix} i = 1,\dots,N & \gamma = 1,\dots, \\ \alpha = 1,\dots,m & \eta = 1,\dots, \end{matrix}$$

The second order Theorem to be proven for this problem is then

Theorem 8.1 Let a_0 be a normal solution to this problem. Then the multipliers of Theorem 6.1 of [2] $\lambda_\rho, \mu_\alpha(t), h_\eta(t), p_i(t), K^\alpha$ $\rho = 0,1,\ldots, p+N+m$; $\alpha = 1,\ldots,m$; $\eta = 1,\ldots,L$; $i = 1,\ldots,N$ can be selected with $\lambda_0 = 1$. If $\lambda_0 = 1$, then these multipliers are unique except for the multipliers $\lambda_{p+N+\alpha}$, $\mu_\alpha(t)$, $p_i(t)$ $(\alpha = 1,\ldots,m;\ i = 1,\ldots,N)$ which are respectively unique up to added terms $c^\alpha, c^\alpha, c^\alpha\psi^\alpha_{x^i}(t)$ with the same arbitrary constants c^α. Furthermore, with H and G as the functions listed above, then for each variation y in the class Y we have that

$$(63)\quad 0 \leq [(p_i(t^c)X^{ic}_{b^\sigma b^\tau})\Big|_{c=0}^{c=1} + G_{b^\sigma b^\tau} + K^\alpha\psi^\alpha_{x^i x^j}(t^0)X^{i0}_{b^\sigma}X^{j0}_{b^\tau}]z^\sigma z^\tau - \int_{t^0}^{t^1}[H_{x^i x^j}y^i y^j +$$

$$2H_{x^i u^h}y^i v^h + H_{u^h u^k}v^h v^k]dt \qquad i,j = 1,\ldots,N, \quad \sigma,\tau = 1,\ldots,r \quad h,k = 1,\ldots,K$$

9. <u>Transformation of the Problem of Section 8</u>

According to [2] the present problem may be re-cast as a problem of the type of section 3 by using the functions

$$(64a)\qquad U^k(t,x,u) \qquad k = 1,\ldots,K$$

as constructed in Lemma 7.1 of [2] on the neighborhood N_1 of $\bar{R}_1$ (the closure of R_1 in R) which satisfy

$$(64b)\quad \theta^\eta(t,x,U(t,x,u)) = 0 \qquad \eta = 1,\ldots,L \text{ on } N_1$$

and by introducing the functions

(65) $\bar{f}^i(t,x,u) \equiv f^i(t,x,U(t,x,u))$, $\bar{L}_\gamma(t,x,u) \equiv L_\gamma(t,x,U(t,x,u))$

$$\bar{\phi}^\alpha(t,x,u) \equiv \phi^\alpha(t,x,U(t,x,u))$$

on N_1. According to the properties of the functions $f^i, L_\gamma, \phi^\alpha$, the functions U^k are C^2 so that also the functions of (65) are C^2 on N_1.

The definition of an arc remains the same as in section 8 and following our convention, we denote arcs in the transformed problem as $\bar{a}$. We now wish to minimize the functional

(66) $$\bar{I}_0(\bar{a}) = g_0(\bar{b}) + \int_{t^0}^{t^1} \bar{L}_0(t,\bar{x},\bar{u})dt$$

subject to the constraints

(67a) $$\dot{\bar{x}}^i = \bar{f}^i(t,\bar{x},\bar{u}) \qquad i = 1,\dots,N$$

(67b) $$\psi^\alpha(t,\bar{x}) \leq 0 \qquad \alpha = 1,\dots,m$$

(67c) $\bar{I}_\gamma(\bar{a}) \leq 0 \qquad \gamma = 1,\dots,p' \qquad \bar{I}_\gamma(\bar{a}) = 0 \qquad \gamma = p'+1,\dots,p$

(67d) $$\bar{x}^i(t^s) = X^{is}(\bar{b}) \qquad i = 1,\dots,N \qquad s = 0,1$$

where

(67e) $$\bar{I}_\gamma(\bar{a}) = g_\gamma(\bar{b}) + \int_{t^0}^{t^1} \bar{L}_\gamma(t,\bar{x},\bar{u})dt \qquad \gamma = 1,\dots,p.$$

By reasoning as in section 8 of [2] and by the correspondence between arcs in the transformed and original problems as

(68a) $\bar{a}$: $\qquad \bar{x}(t) \qquad \bar{u}(t) \qquad \bar{b} \qquad t^0 \leq t \leq t^1$

and

(68b) a: $x(t) = \bar{x}(t) \qquad u(t) \equiv U(t,\bar{x}(t),\bar{u}(t)) \qquad b = \bar{b} \qquad t^0 \le t \le t^1$

and by the fact that the function U evaluated along a_0 satisfies

$$(69) \qquad U(t,x_0(t),u_0(t)) = u_0(t) \qquad t^0 \le t \le t^1$$

we see that the arc $\bar{a}_0$ defined as

$$(70) \quad \bar{a}_0: \ \bar{x}_0(t) = x_0(t) \qquad \bar{u}_0(t) = u_0(t) \qquad \bar{b}_0 = b_0 \qquad t^0 \le t \le t^1$$

(where the unbarred quantites are from the arc a_0 of (58a)), will be a solution to the transformed problem if the arc a_0 of (58a) is a solution to the original problem.

Thus according to Theorem 3.2 of [1] there are multipliers $\bar{K}^\alpha$, $\bar{\lambda}_\rho$, $\bar{p}_i(t)$, $\bar{\mu}_\alpha(t)$, $\rho = 0,1\ldots,p + N + m$; $i = 1,\ldots,N$; $\alpha = 1,\ldots,m$; and functions $\bar{H}$ and $\bar{G}$ where by section 8 of [2] the relationship between these terms and the unbarred terms of Theorem 6.1 of [2] are

$$(71) \quad \bar{H}(t,x,u,p(t),\mu(t)) = H(t,x,U(t,x,u),p(t),\mu(t),h(t)) + h_\eta(t)\theta^\eta(t,x,U(t,x,u)$$

$$\bar{G} = G , \qquad \bar{\lambda}_\rho = \lambda_\rho , \qquad \bar{p}_i(t) = p_i(t) , \qquad \bar{\mu}_\alpha(t) = \mu_\alpha(t) , \qquad \bar{K}^\alpha = K^\alpha$$

$$\rho = 0,1,\ldots,p + N + m , \qquad i = 1,\ldots,N \qquad \alpha = 1,\ldots,m$$

In order to prove Theorem 8.1 we need the following definitions and results. Corresponding to a variation of the solution arc for the problem of section 3 as described in (18) we define a variation $\overline{\delta a}$ of $\bar{a}_0$ for the transformed problem as

$$(72) \quad \bar{\bar{\delta a}}: \qquad \bar{\bar{\delta x}}(t) \qquad \bar{\bar{\delta u}}(t) \qquad \bar{\bar{\delta b}} \qquad t^0 \le t \le t^1$$

We also define functionals $\bar{J}_\rho$ $(\rho = 0,1\ldots,p + 2N)$ in a directly analagous fashion to that used in defining the functionals of (19) for the problem of section 3.

We now prove

Lemma 9.1 Given a variation δa of a_0 as in (59) satisfying (61d), then one can construct a variation $\bar{\bar{\delta a}}$ of $\bar{a}_0$

$$(73) \quad \bar{\bar{\delta a}}: \qquad \bar{\bar{\delta x}}(t) = \delta x(t) \qquad \bar{\bar{\delta u}}(t) \qquad \bar{\bar{\delta b}} = \delta b \qquad t^0 \le t \le t^1$$

(where $\delta x(t)$, δb are the quantities of (59)) with the property that

$$(74) \qquad \bar{\delta U}^k = U^k_{x^i}\bar{\bar{\delta x}}^i + U^k_{u^h}\bar{\bar{\delta u}}^h = \delta u^k \qquad h,k = 1,\ldots,k \qquad i = 1,\ldots,N$$

(where δu^k are the functions of (59)).

Proof: In the proof of this lemma all derivatives will be evaluated along a_0 unless otherwise specified. In order to prove this lemma, we notice that by differentiating equation (64b) we have

$$(75a) \qquad \theta^\eta_{x^i} + \theta^\eta_{u^k}U^k_{x^i} = 0$$

$$(75b) \qquad \theta^\eta_{u^k}U^k_{u^h} = 0$$

$$t^0 \le t \le t^1 \qquad \eta = 1,\ldots,L \qquad i = 1,\ldots,N \qquad h,k = 1,\ldots,K$$

Furthermore, by (61d) and (73) we also have that

$$(75c) \qquad \theta^\eta_{x^i}\bar{\bar{\delta x}}^i + \theta^\eta_{u^k}\delta u^k = 0$$

so that

(75d) $\theta^{\eta}_{x^i}\bar{\bar{\delta x}}^i = -\theta^{\eta}_{u^k}\delta u^k \qquad t^0 \le t \le t^1 \qquad \eta = 1,\ldots,L \qquad i = 1,\ldots,N \qquad h = 1,\ldots,K$

Using these results and for the moment defining $\bar{\bar{\delta u}}^k$ $(k = 1,\ldots,K)$ arbitrarily we see that with $\bar{\delta}U^k$ and δu^k as the quantities respectively of (74) and (59) then

(76) $\theta^{\eta}_{u^k}[\bar{\delta}U^k - \delta u^k] = \theta^{\eta}_{u^k}[U^k_{x^i}\bar{\bar{\delta x}}^i + U^k_{u^h}\bar{\bar{\delta u}}^h - \delta u^k] = -\theta^{\eta}_{x^i}\bar{\bar{\delta x}}^i + 0 + \theta^{\eta}_{x^i}\bar{\bar{\delta x}}^i = 0$

$$t^0 \le t \le t^1 \qquad \eta = 1,\ldots,L \qquad i = 1,\ldots,N$$

so that (76) is true no matter how we define $\bar{\bar{\delta u}}^k$ $(k = 1,\ldots,K)$ $\qquad h,k = 1,\ldots,K.$

Furthermore by (37-2) of [2] the matrix

$$(U^k_{u^h}(t)) = \begin{bmatrix} U^1_{u^1}, & U^1_{u^2}, & \ldots, & U^1_{u^K} \\ U^2_{u^1}, & U^2_{u^2}, & \ldots, & U^2_{u^K} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ U^K_{u^1}, & U^K_{u^2}, & \ldots, & U^K_{u^K} \end{bmatrix}$$

has rank $K - L$ on $[t^0 t^1]$ so that at any time $\bar{t}$, $(t^0 \le \bar{t} \le t^1)$ we can select $K - L$ linearly independent columns, say columns $i_1,\ldots,i_{K-L}$

(78) $$U^k_{u^{i_1}}(\bar{t}),\ldots,U^k_{u^{i_{K-L}}}(\bar{t}) \qquad k = 1,\ldots,K$$

and denote the resulting $K \times K\text{-}L$ matrix as $M(\bar{t})$. Then by continuity, the linear independence of the columns of M persists on some neighborhood $(\bar{t}-\bar{\Delta},\bar{t}+\bar{\Delta})$.

Now consider the equations

$$(79)\quad U^k_{u^{i_j}} U^k_{u^{i_\ell}} w^\ell + U^k_{u^{i_j}} [U^k_{x^s} \overline{\delta x}^s + U^k_{u^{\alpha_r}} v^r - \delta u^k] = 0 \qquad (\bar{t} - \bar{\Delta},\ \bar{t} + \bar{\Delta})$$

$$\alpha_r \neq i_\ell,\ i_j \qquad 1 \le \alpha_r, i_\ell, i_j \le K \quad r = 1,\ldots,L, \quad \ell, j = 1,\ldots,K-L$$

$$s = 1,\ldots,N \qquad k = 1,\ldots,K$$

where: i) $i_1,\ldots,i_{K-L}$ are the indices of (78), ii) the unlisted argument in all terms except w is t and iii) v indicates an arbitrary L dimensional vector in time which is continuous.

According to the definition of M, the coefficient of the w vector is $M^T M$ and so has rank $K - L$ on $(\bar{t} - \bar{\Delta},\ \bar{t} + \bar{\Delta})$. Then there is the piecewise continuous solution in time to (79)

$$(80)\quad w^\ell = w^\ell(t, \overline{\delta x}(t), v(t), \delta u(t)) \qquad (\bar{t} - \bar{\Delta},\ \bar{t} + \bar{\Delta}) \qquad \ell = 1,\ldots,K - L$$

where $\bar{\Delta}$ is the constant selected above. With v^h still arbitrary, define

$$(81)\quad \overline{\delta u}^{\alpha_h} = v^h \qquad h = 1,\ldots,L; \qquad \alpha_h \neq i_1,\ldots,i_{K-L}$$

$$\overline{\delta u}^{i_\ell} = w^\ell \qquad \ell = 1,\ldots,K - L \qquad (\bar{t} - \bar{\Delta},\ \bar{t} + \bar{\Delta})$$

Then by re-writing (79), the solution (80) yields

$$(82)\quad M^T[U^k_{x^s} \overline{\delta x}^s + U^k_{u^h} \overline{\delta u}^h - \delta u^k] = 0 \qquad (\bar{t} - \bar{\Delta},\ \bar{t} + \bar{\Delta}), \qquad h,k = 1,\ldots,K,$$

$$s = 1,\ldots,N$$

where M is the matrix previously defined. Now including the equation (76) and recognizing that the bracketed term of (82) is just $\bar{\delta U}^k - \delta u^k$ as defined in (74 and (59) we have by (82) that

$$(83) \quad \theta^{\eta}_{u^k}[\bar{\delta U}^k - \delta u^k] = 0 \qquad M^T[\bar{\delta U}^k - \delta u^k] = 0 \qquad (\bar{t} - \bar{\Delta}, \bar{t} + \bar{\Delta})$$

Since by (75b) and our selection of $\bar{\Delta}$ the columns of M are each perpendicular to the rows of the matrix $(\theta^{\eta}_{u^k})$ on $(\bar{t} - \bar{\Delta}, \bar{t} + \bar{\Delta})$ and are themselves linearly independent, then by the properties of tne matrix $(\theta^{\eta}_{u^k})$, we must have that

$$(84) \quad \bar{\delta U}^k(t) - \delta u^k(t) = 0 \qquad (\bar{t} - \bar{\Delta} \quad \bar{t} + \bar{\Delta}) \qquad k = 1,\ldots,K.$$

Now we can do this construction for each point t on $[t^0,t^1]$ (1) and so get a neighborhood N_t of t for which the Lemma is true. Then by compactness we can cover $[t^0,t^1]$ by a finite number of the resulting neighborhoods N_{t_j} and hence by a finite number of closed neighborhoods $\bar{N}_{t_j}$ which touch only at end points. Then calling $\bar{\delta u}_j$ the functions as constructed above on N_{t_j} and setting

$$\bar{\delta u} = \bar{\delta u}_j \quad \text{on} \quad \bar{N}_{t_j}$$

the Lemma is proven.

(1) If $t = t^0$ or t^1 then our neighborhood is one sided.

As a next result, we prove

<u>Lemma 9.2</u> Normality of the problem of section 8 implies normality of the transformed problem.

Using the variations $\delta_\omega a$ $(\omega = 1,\dots,\Omega)$ defined above (61) and satisfying (61) then we use Lemma 9.1 to define Ω [(1)] variations $\bar{\delta}_\omega \bar{a}$ of $\bar{a}_0$

$$(85) \quad \bar{\delta}_\omega \bar{a}: \quad \bar{\delta}_\omega \bar{x}(t) = \delta_\omega x(t) \ , \qquad \bar{\delta}_\omega \bar{u}(t) \ , \qquad \bar{\delta}_\omega \bar{b} = \delta_\omega b \qquad t^0 \le t \le t^1$$

$$\omega = 1,\dots,\Omega$$

satisfying

$$(86) \quad \bar{\delta}_\omega U^k \equiv U^k_{x^i} \bar{\delta}_\omega \bar{x}^i + U^k_{u^h} \bar{\delta}_\omega \bar{u}^h = \delta_\omega u^k \qquad k = 1,\dots,K \qquad \omega = 1,\dots,\Omega$$

With this construction and by (65), we see that

$$(87a) \quad \bar{\delta}_\omega \dot{\bar{x}}^i = \delta_\omega \dot{x}^i = f^i_{x^j} \delta_\omega x^j + f^i_{u^k} \delta_\omega u^k = f^i_{x^j} \bar{\delta}_\omega \bar{x}^j + f^i_{u^k} (U^k_{x^i} \bar{\delta}_\omega \bar{x}^i + U^k_{u^h} \bar{\delta}_\omega \bar{u}^h) = \bar{\delta}_\omega \bar{f}^i$$

$$(87b) \quad \bar{\delta}_\omega \psi^\alpha = \psi^\alpha_{x^i} \bar{\delta}_\omega \bar{x}^i = \psi^\alpha_{x^i} \delta_\omega x^i = \delta_\omega \psi^\alpha \qquad \alpha = 1,\dots,m$$

$$(87c) \quad \bar{\delta}_\omega \bar{L}_\gamma = L_{\gamma x^i} \bar{\delta}_\omega \bar{x}^i + L_{\gamma u^k} [U^k_{x^i} \bar{\delta}_\omega \bar{x}^i + U^k_{u^h} \bar{\delta}_\omega \bar{u}^h] = L_{\gamma x^i} \delta_\omega x^i + L_{\gamma u^k} \delta_\omega u^k = \delta_\omega L_\gamma$$

$$\gamma = 0,\dots,p \qquad \omega = 1,\dots,\Omega \qquad j,i = 1,\dots,N, \qquad h,k = 1,\dots,K \qquad t^0 \le t \le t$$

(1) The term Ω is the constant referred to above (61).

so that with $\bar{J}_\rho$ as the functionals introduced below (72) then

(87d) $$\bar{J}'_\gamma(\bar{a}_0,\bar{\delta}_\omega\bar{a}) = g_{\gamma_{b^\sigma}}\bar{\delta}_\omega\bar{b}^\sigma + \int_{t^0}^{t^1}\bar{\delta}_\omega\bar{L}_\gamma\,dt = g_{\gamma_{b^\sigma}}\delta_\omega b^\sigma + \int_{t^0}^{t^1}\delta_\omega L_\gamma\,dt = J'_\gamma(a_0,\delta_\omega a)$$

$$\gamma = 0,1,\ldots,p$$

and

(87e) $$\bar{J}'_{p+i}(\bar{a}_0,\bar{\delta}_\omega\bar{a}) = \bar{\delta}_\omega\bar{x}^i(t^1) - X^{i1}_{b^\sigma}\bar{\delta}_\omega\bar{b}^\sigma = \delta_\omega x^i(t^1) - X^{i1}_{b^\sigma}\delta_\omega b^\sigma = J'_{p+i}(a_0,\delta_\omega a)$$

(87f) $$\bar{J}'_{p+N+i}(\bar{a}_0,\bar{\delta}_\omega\bar{a}) = X^{i0}_{b^\sigma}\bar{\delta}_\omega\bar{b}^\sigma - \bar{\delta}_\omega\bar{x}^i(t^0) = X^{i0}_{b^\sigma}\delta_\omega b^\sigma - \delta_\omega x^i(t^0) = J'_{p+N+i}(a_0,\delta_\omega a)$$

$$i = 1,\ldots,N \qquad \omega = 1,\ldots,\Omega \qquad \sigma = 1,\ldots,r$$

Furthermore, according to our definitions of the arc $\bar{a}_0$ and $\bar{I}_\gamma$ we see that

(88) $$\bar{I}_\gamma(\bar{a}_0) = I_\gamma(a_0) \qquad \gamma = 0,1,\ldots,p$$

so that the indices γ_k defined for the transformed problem as in (21) agree with the indices γ_k defined for the problem of section 8. Then by (71) and (87) together with the properties of the variations $\delta_\omega a$ defined above (61) the construction of $\bar{a}_0$ and by using the definition of normality for the transformed problem, as given in section 4, we see that the transformed problem is normal, this proving the lemma.

Next, corresponding to the class of variations defined in (24), define the class $\bar{Y}$ of variations $\bar{y}$ of $\bar{a}_0$

(89) $\bar{\mathcal{Y}}$: $\bar{y}^j(t)$ $\bar{v}^k(t)$ $\bar{z}^\sigma$ $t^0 \le t \le t^1$

$j = 1,\ldots,N$ $k = 1,\ldots,K$ $\sigma = 1,\ldots,r$

which satisfies

(90a) $\dot{\bar{y}}^i = \bar{\delta}_{\bar{\mathcal{Y}}}\bar{f}^i$ $i = 1,\ldots,N$

(90b) $\bar{\delta}_{\bar{\mathcal{Y}}}\psi^\alpha(t) \le 0$ on a neighborhood of S^α $\alpha = 1,\ldots,m$

(90c) $\bar{\delta}_{\bar{\mathcal{Y}}}\psi^\alpha(t^c) = 0$ if $\bar{\bar{\mu}}_\alpha(t^c) \ne \bar{\lambda}_{p+N+\alpha}$ $c = 0,1$ $\alpha = 1,\ldots,m$

(90d) $\bar{\delta}_{\bar{\mathcal{Y}}}\psi^\alpha(\bar{t}) = 0$ if $\bar{\bar{\mu}}_\alpha(t)$ is not constant on a neighborhood of $\bar{t}$

(90e) $\bar{\delta}_{\bar{\mathcal{Y}}}\psi^\alpha(t^o) = 0$ if $\psi^\alpha(t^o) = 0$ $\alpha = 1,\ldots,m$

(90f) $\bar{J}'_\gamma(\bar{a}_0,\bar{\mathcal{Y}}) = 0$ if $\bar{\lambda}_\gamma \ne 0$ $1 \le \gamma \le p'$

(90g) $\bar{J}'_\gamma(\bar{a}_0,\bar{\mathcal{Y}}) \le 0$ if $\bar{\lambda}_\gamma = 0$ $\gamma \ne \gamma_k$ $1 \le \gamma \le p'$

(90h) $\bar{J}'_\rho(\bar{a}_0,\bar{\mathcal{Y}}) = 0$ $p' < \rho \le p + 2N$

where $\bar{\mu}_\alpha(t)$, $\bar{\lambda}_{p+N+\alpha}$, $\bar{\lambda}_\gamma$ are multipliers of Theorem 6.1 of [2] and γ_k are the indices referred to below (88).

The next lemma is proven in a directly analagous manner to that used in constructing the variations $\bar{\delta}_\omega\bar{a}$ of Lemma 9.2 and the steps will not be repeated.

Lemma 9.3 For each variation $\mathcal{Y}$ in the class Y defined in (62) there is a variation $\bar{\mathcal{Y}}$ in the class $\bar{Y}$.

Proof: For the variation $\mathcal{Y}$

(91) $\mathcal{Y}$: $\quad y^i(t) \quad v^k(t) \quad z^\sigma \quad t^0 \le t \le t^1$

in Y, the corresponding variation in $\bar{Y}$ is

(92) $\bar{\mathcal{Y}}$: $\quad \bar{y}^i(t) = y^i(t) \quad \bar{v}^k(t) \quad \bar{z}^\sigma = z^\sigma \quad t^0 \le t \le t^1$

satisfying

(93) $$\bar{\delta}_{\bar{\mathcal{Y}}} U^k = U^k_{x^i} \bar{y}^i + U^k_{u^h} \bar{v}^h = v^k \qquad k = 1,\dots,K$$

where this variation exists by Lemma 9.1.

10. Proof of Theorem 8.1

With the results of section 9 we may prove:

Lemma 10.1 Theorem 4.1 implies Theorem 8.1

According to Theorem 4.1 applied to the transformed problem together with the relation between the multipliers of the original and transformed problems as listed in (71), we see that the first statement of Theorem 4.1 implies the first statement of Theorem 8.1.

In order to prove the second statement of Theorem 8.1, we notice that by the above reasoning, if $\lambda_0 = 1$ then the multipliers K^α $(\alpha = 1,\dots,m)$; $\lambda_0,\dots,\lambda_{p+N}$ are unique while the multipliers $\mu_\alpha(t)$, $\lambda_{p+N+\alpha}$ $(\alpha = 1,\dots,m)$ are unique up to arbitrary constants c^α and with these same constants, the multipliers $p_i(t)$ $(i = 1,\dots,N)$ are unique up to added terms $c^\alpha \psi^\alpha_{x^i}(t)$. However for the present problem, there are the additional multipliers

(94) $\quad h_\eta(t) \quad \eta = 1,\dots,L \quad t^0 \le t \le t^1$

and we must show that these are also unique when $\lambda_0 = 1$.

Now by writing out the third relation of (31) of [2]

$$(95) \qquad H_{u^k} = 0 \qquad t^0 \le t \le t^1 \qquad k = 1,\dots,K$$

where H is the function of Theorem 6.1 of [2] and designating

$$(96) \qquad p_i(t) = \hat{p}_i(t) + c^\alpha \psi^\alpha{}_i(t) \quad \text{and} \quad \mu_\alpha(t) = \hat{\mu}_\alpha(t) + c^\alpha$$

where the "capped" portion of the terms indicates the unique part of the multipliers as described above then

$$(97a) \qquad 0 = H_{u^k} = (\hat{p}_i(t) + c^\alpha \psi^\alpha{}_{x^i}(t)) f^i_{u^k} - \lambda_0 L_{0\,u^k} - \lambda_\gamma L_{\gamma\,u^k} - (\hat{\mu}_\alpha(t) + c^\alpha)\phi^\alpha_{u^k} - h_\eta(t)\theta^\eta_{u}$$

$$i = 1,\dots,N \qquad k = 1,\dots,K \qquad \alpha = 1,\dots,m \qquad \gamma = 1,\dots,p \qquad \eta = 1,\dots,L$$

Then using the definition of ϕ^α given below (58b) we see that along a_0 the arbitrary parts of H_{u^k} cancel out leaving

$$(97b) \qquad 0 = H_{u^k} = \hat{p}_i(t) f^i_{u^k} - \lambda_0 L_{0\,u^k} - \lambda_\gamma L_{\gamma\,u^k} - \hat{\mu}_\alpha(t)\phi^\alpha_{u^k} - h_\eta(t)\theta^\eta_{u^k}$$

Thus by the statement involving (58b) we see that also the multipliers $h_\eta(t)(\eta = 1,\dots,L)$ are fixed and statement 2 of Theorem 8.1 is proven.

In order now to prove the last statement of Theorem 8.1, let y be a variation in the class Y defined in (62) and by Lemma 9.1 let $\bar{y}$ be the correspon variation in the class $\bar{Y}$ as constructed in (92). By Theorem 4.1 and the relations between the multipliers as listed in (71) we have

$$(98)\quad 0 \le [(p_i(t^c)X^{ic}_{b^\sigma b^\tau})\Big|_{c=0}^{c=1} + G_{b^\sigma b^\tau} + K^\alpha \psi^\alpha_{x^i x^j}(t^0)X^{i0}_{b^\sigma}X^{j0}_{b^\tau}]z^\sigma z^\tau$$

$$- \int_{t^0}^{t^1} [\bar{H}_{x^i x^j}\bar{y}^i\bar{y}^j + 2\bar{H}_{x^i u^h}\bar{y}^i\bar{v}^h + \bar{H}_{u^h u^k}\bar{v}^h\bar{v}^k]dt$$

$$i,j = 1,\dots,N \qquad \sigma,\tau = 1,\dots,r \qquad h,k = 1,\dots,K$$

where $\bar{H}$ is the function of Theorem 4.1 (or equivalently of Theorem 3.2 of [1]) referred to in (71).

Now, differentiating the relations (64b) we obtain

$$(99)\qquad \frac{\partial\theta^\eta}{\partial x^i}(t,x,U(t,x,u)) = 0 \qquad \frac{\partial\theta^\eta}{\partial u^h}(t,x,U(t,x,u)) = 0$$

$$\frac{\partial^2\theta^\eta}{\partial x^i\partial x^j}(t,x,U(t,x,u)) = 0 \qquad \frac{\partial^2\theta^\eta}{\partial x^i\partial u^k}(t,x,U(t,x,u)) = 0$$

$$\frac{\partial^2\theta^\eta}{\partial u^h\partial u^k}(t,x,U(t,x,u)) = 0 \qquad \eta = 1,\dots,L \quad i,j = 1,\dots,N \quad h,k = 1,\dots,K$$

where the derivatives in (99) are taken on the region of definition N_1 of the functions U and so include the arc a_0 .

Next, by (71) we have

$$(100a)\qquad \bar{H}_{x^i} = H_{x^i} + H_{u^h}U^h_{x^i} + h_\eta(t)\frac{\partial\theta^\eta}{\partial x^i}(t,x,U(t,x,u))$$

$$(100b)\qquad \bar{H}_{x^i x^j} = H_{x^i x^j} + H_{x^i u^h}U^h_{x^j} + H_{u^h x^j}U^h_{x^i} + H_{u^h u^r}U^h_{x^i}U^r_{x^j} + H_{u^h}U^h_{x^i x^j} +$$

$$h_\eta(t)\frac{\partial^2\theta^\eta}{\partial x^i\partial x^j}(t,x,U(t,x,u))$$

(100c) $$\bar{H}_{x^i u^k} = H_{x^i u^r} U^r_{u^k} + H_{u^h u^r} U^h_{x^i} U^r_{u^k} + H_{u^h} U^h_{x^i u^k} + h_\eta(t) \frac{\partial^2 \theta^\eta}{\partial x^i \partial u^k}(t,x,\ U(t,x,u))$$

(100d) $$\bar{H}_{u^k} = H_{u^r} U^r_{u^k} + h_\eta(t) \frac{\partial \theta^\eta}{\partial u^k}(t,x,U(t,x,u))$$

(100e) $$\bar{H}_{u^k u^h} = H_{u^r u^s} U^r_{u^k} U^s_{u^h} + H_{u^r} U^r_{u^k u^h} + h_\eta(t) \frac{\partial^2 \theta^\eta}{\partial u^k \partial u^h}(t,x,U(t,x,u))$$

$$i,j = 1,\ldots,N; \qquad s,h,r,k = 1,\ldots,K; \qquad \eta = 1,\ldots,L$$

where we have used the notation H_{x^i} to denote $\frac{\partial H}{\partial x^i}(t,x,u)$, i.e. not including the contribution of the function $U(t,x,u)$.

Now according to Theorem 6.1 of [2], we have along a_0 that

(101) $$H_{u^h} = 0$$

and furthermore by (99) that all derivatives of θ^η appearing in (100) vanish along $\bar{a}_0$ so that we may delete from (100) all terms involving these quantities. Then using (100) together with the variation $\bar{y}$ of (92) the integrand of (98) is

(102) $$[H_{x^i x^j} + 2H_{x^i u^h} U^h_{x^j} + H_{u^h u^r} U^h_{x^i} U^r_{x^j}] y^i y^j + 2[H_{x^i u^r} U^r_{u^k} + H_{u^h u^r} U^h_{x^i} U^r_{u^k}] y^i \bar{v}^k$$

$$+ H_{u^r u^s} U^r_{u^k} U^s_{u^h} \bar{v}^k \bar{v}^h \qquad i,j = 1,\ldots,N; \qquad r,s,h,k = 1,\ldots,K$$

Using (92) and (93) we see that

(103) $$2H_{x^i u^h}[U^h_{x^j} y^j + U^h_{u^k} \bar{v}^k] y^i = 2H_{x^i u^h} y^i \bar{\delta}_{\bar{y}} U^h = 2H_{x^i u^h} y^i v^h$$

$$(104)\quad H_{u^h u^r}[U^h_{x^i} U^r_{x^j} y^i y^j + 2U^h_{x^i} U^r_{u^k} y^i \bar{v}^k + U^r_{u^k} U^h_{u^s} \bar{v}^k \bar{v}^s] =$$

$$H_{u^h u^r}(U^h_{x^i} y^i + U^h_{u^s} \bar{v}^s)\ (U^r_{x^j} y^j + U^r_{u^k} \bar{v}^k)$$

$$= H_{u^h u^r} \bar{\delta}_{\bar{y}} U^h \bar{\delta}_{\bar{y}} U^r = H_{u^h u^r} v^h v^r$$

Then by substituting (103) and (104) into (102) we see that the integrand of (98) is just

$$(105)\quad H_{x^i x^j} y^i y^j + 2H_{x^i u^h} y^i v^h + H_{u^h u^r} v^h v^r \qquad i,j = 1,\dots,N, \qquad h,r = 1,\dots,K$$

which is the required integrand of Theorem 8.1. Then the inequality of Theorem 8.1 is proven. Thus Lemma 10.1 and hence also Theorem 8.1 are proven.

11. A Problem With Equality and Inequality State and Differential Constraints

Our next generalization leads us to the problem of section 8 in which we have adjoined additional constraints of the form.

$$(105)\qquad \psi^{\alpha}(t,x) = 0 \qquad \theta^{\eta}(t,x,u) \le 0$$

Thus with arcs a as in section 8 we wish to minimize the functional

$$(107)\qquad I_0(a) = g_0(b) + \int_{t^0}^{t^1} L_0(t,x,u)dt$$

among all arcs which satisfy the conditions

$$(108a)\qquad \dot{x}^i = f^i$$

(108b) $\psi^{\alpha}(t,x) \leq 0 \qquad \alpha = 1,\ldots,m' \qquad \psi^{\alpha}(t,x) = 0 \qquad \alpha = m'+1,\ldots,m$

(108c) $\theta^{\eta}(t,x,u) \leq 0 \qquad \eta = 1,\ldots,L' \qquad \theta^{\eta}(t,x,u) = 0 \qquad \eta = L'+1,\ldots,L$

(108d) $I_{\gamma}(a) \leq 0 \qquad \gamma = 1,\ldots,p' \qquad I_{\gamma}(a) = 0 \qquad \gamma = p'+1,\ldots,p$

(108e) $x^{i}(t^{s}) = X^{is}(b) \qquad i = 1,\ldots,N \qquad s = 0,1$

where $\psi^{\alpha},\phi^{\alpha},I_{\gamma},f^{i},\theta^{\eta},L_{\gamma},g_{\gamma},X^{is}(\alpha = 1,\ldots,m'; \quad \gamma = 0,1,\ldots,p; \quad i = 1,\ldots,N;$ $\eta = L'+1,\ldots,L \quad s = 0,1)$ are defined as in section 8 and in addition, the functions $\theta^{\eta},\psi^{\alpha}$, $\phi^{\alpha} = \psi^{\alpha}_{t} + \psi^{\alpha}_{x^{i}}f^{i}$ for $\alpha = m'+1,\ldots,m$ and $\eta = 1,\ldots,L'$ are of class $C^{2}C^{3}$ and C^{2} respectively on R.

Now corresponding to (58a) assume that the arc

(109a) $a_{0}: \qquad x_{0}(t) \qquad u_{0}(t) \qquad b_{0} \qquad t^{0} \leq t \leq t^{1}$

with $u_{0}(t)$ continuous, is a solution to the present problem.

The sets R, R_{0}, R_{1} are defined here analagously to section 8 and corresponding to the assumption concerning the matrix of (58b) we assume that the matrix

(109b) $$\begin{bmatrix} \theta^{\eta}_{u^{k}} & \delta_{\eta\rho}\theta^{\rho} & 0 \\ \phi^{\alpha}_{u^{k}} & 0 & \delta_{\alpha\beta}\psi^{\beta} \end{bmatrix} \qquad \begin{array}{l} \eta,\rho = 1,\ldots,L \\ \alpha,\beta = 1,\ldots,m \\ k = 1,\ldots,K \end{array}$$

has rank $L + m$ on $\bar{R}_{1}$, the closure of R_{1} in R .

Now according to the above, we have by the definition of the functions ϕ^α, that the matrices

$$(109c) \qquad (\phi^\alpha_{u^k} \quad \delta_{\alpha\beta}\psi^\beta) = (\psi^\alpha_{x^i} f^i_{u^k} \quad \delta_{\alpha\beta}\psi^\beta) \qquad \alpha,\beta = 1,\ldots,m \quad i = 1,\ldots,N \quad k = 1,\ldots,K$$

have rank m on $\bar{R}_1$. The left most terms of this last matrix are the product of the matrices

$$(109d) \qquad (\psi^\alpha_{x^i}) \quad (f^i_{u^k}) \qquad \alpha = 1,\ldots,m \quad k = 1,\ldots,K \quad i = 1,\ldots,N$$

The above rank condition applied at t^0 thus implies that for those indices β_j between 1 and m such that

$$\psi^\beta(t^0) = 0$$

we have (with $\tilde{m}$ as the number of such indices) that the matrix

$$\left(\psi^{\beta_j}_{x^i}(t^0)\right) \qquad j = 1,\ldots,\tilde{m}$$

has rank $\tilde{m}$. Thus in particular the matrix

$$(109e) \qquad \left(\psi^{m'+\beta}_{x^i}(t^0)\right) \qquad \beta = 1,\ldots,m - m'$$

has rank $m - m'$.

We make one additional assumption here which will be useful in our construction, namely that if $\alpha_j \quad j = 1,\ldots,\hat{n}$ are the indices between 1 and m' for which both

$$(109f) \qquad \psi^\alpha(t^0) \neq 0 \quad \text{and} \quad \mu_\alpha(t^0) \neq \lambda_{p+N+\alpha}$$

where $\mu_\alpha(t)$, $\lambda_{p+N+\alpha}$ are terms of Theorem 9.1 of [2] and γ_s $(s=1,\ldots,\bar{\bar{m}})$ are the indices between 1 and m' for which

$$\psi^{\gamma}(t^0) = 0 \tag{109g}$$

then the last $m - m'$ rows of the matrix

$$\begin{bmatrix} \psi^{\alpha_j}_{x^i}(t^0) \\ \\ \psi^{\gamma_s}_{x^i}(t^0) \\ \\ \psi^{m'+\beta}_{x^i}(t^0) \end{bmatrix} \qquad \begin{array}{ll} s = 1,\ldots,\bar{\bar{m}} & \\ j = 1,\ldots,\hat{n} & \beta = 1,\ldots,m - m' \\ i = 1,\ldots,N & \end{array} \tag{109h}$$

are linearly independent of the other rows. This condition essentially says that the gradients of the functions $\psi^{m'+\beta}(t^0)$ $(\beta=1,\ldots, m - m')$ are linearly independent of the gradients of $\psi^{\gamma_s}(t^0)$ $(s - 1,\ldots,\bar{\bar{m}})$ and ψ^{α_j} $(j=1,\ldots,\hat{n})$. Notice that by the condition on the matrix above (109e) we already have that the gradients of $\psi^{m'+\beta}(t^0)$ are linearly independent of the gradients of $\psi^{\gamma_s}(t^0)$.

We next add a similar "extra statement" to Theorem 9.1 of [2] for this problem which corresponds to the "extra statement" added to Theorem 6.1 of [2] previously.

With λ_ρ, K^α, $p_i(t)$ as multipliers of Theorem 9.1 of [2], then for the case when

$$\lambda_\gamma = 0 \text{ if } \gamma \notin L \quad (0 \le \gamma \le p) \quad K^\alpha = 0 \quad \lambda_{p+N+\alpha} = \mu_\alpha(\bar{t}) = s^\alpha$$

$$\alpha = 1,\ldots,m$$

and

$$p_j(\bar{t}) = s^\alpha \psi^\alpha_{x^j}(\bar{t}) \quad j \notin K\ , \quad p_i(t_i) = s^\alpha \psi^\alpha_{x^i}(t_i) \quad i \varepsilon K$$

for any points $\bar{t}$, t_i in $[t^0, t^1]$ and any constants s^α $(\alpha = 1,\ldots,m)$ and where K, L are the respective sets of indices $1 \le i \le N$, $0 \le \gamma \le p$ for which $f^i \equiv L_\gamma \equiv 0$ then we must have that

$$p_i(t) = s^\alpha \psi^\alpha_{x^i}(t) \qquad t^0 \le t \le t^1 \qquad i = 1,\ldots,N$$

This statement is proven in the same manner that (36) of [2] was proven.

Normality for the present problem is defined in an analagous fashion to that used in section 8.

For this problem, define the analagous functionals J_ρ and their variations as in section 8 and also let γ_k $(k = 1,\ldots,\bar{p})$ be defined in an analagous fashion. Let $\mu_\alpha(t), \lambda_{p+N+\alpha}$ be the terms of Theorem 9.1 of [2] and let

(110) $\delta\alpha$: $\qquad \delta x(t) \qquad \delta u(t) \qquad \delta b \qquad t^0 \le t \le t^1$

denote a variation of the arc a_0 of (109a). Then we call the arc a_0 a normal solution to our problem if there are

$$\Omega = p - \bar{p} + 2N \tag{111}$$

variations $\delta_\omega a$ $(\omega = 1,\ldots,\Omega)$ which satisfy the conditions

$$\delta_\omega \dot{x}^i = \delta_\omega f^i \qquad i = 1,\ldots,N \tag{112a}$$

(112b) $\qquad \delta_\omega \psi^\alpha(t^0) = 0 \quad \text{if} \quad \mu_\alpha(t^0) \neq \lambda_{p+N+\alpha} \qquad \alpha = 1,\ldots,m'$

(112c) $\qquad \delta_\omega \psi^\alpha(t) = 0 \quad \text{on a neighborhood of} \quad S^\alpha \qquad \alpha = 1,\ldots,m'$

(112d) $\qquad \delta_\omega \psi^\alpha(t) = 0 \qquad t^0 \leq t \leq t^1 \qquad \alpha = m' + 1,\ldots,m$

(112e) $\qquad \delta_\omega \theta^\eta(t) = 0 \qquad t^0 \leq t \leq t^1 \qquad \eta = L' + 1,\ldots,L$

(112f) $\qquad \delta_\omega \theta^\eta(t) = 0 \quad \text{on a neighborhood of} \quad p^\eta \qquad \eta = 1,\ldots,L'$

where $S^\alpha \equiv \{t|\psi^\alpha(t) = 0\}$ $\;1 \leq \alpha \leq m'$; $\;p^\eta \equiv \{t|\theta^\eta(t) = 0\}$ $\quad 1 \leq \eta \leq L'$

in addition to providing that the matrix

(112g) $\qquad (J'_\rho(a_0,\delta_\omega,a)) \qquad \rho = 1,\ldots,p + 2N \qquad \rho \neq \gamma_k \qquad \omega = 1,\ldots,\Omega$

is non-singular.

Next, define the class Y of variations

y: $\qquad y^i(t) \quad v^k(t) \quad z^\sigma \quad t^0 \leq t \leq t^1$

of a_0 which satisfy the conditions

(113a) $\qquad \dot{y}^i = \delta_y f^i \qquad i = 1,\ldots,N$

(113b) $\qquad \delta_y \psi^\alpha(t) \leq 0 \quad \text{on a neighborhood of} \quad S^\alpha \qquad \alpha = 1,\ldots,m'$

(113c) $\qquad \delta_y \psi^\alpha(t^c) = 0 \quad \text{if} \quad \mu_\alpha(t^c) \neq \lambda_{p+N+\alpha} \qquad c = 0,1 \qquad \alpha = 1,\ldots,m'$

(113d) $\qquad \delta_y \psi^\alpha(\bar{t}) = 0$ if $\mu_\alpha(t)$ is not constant on a neighborhood of $\bar{t}$ $\qquad \alpha = 1,\ldots,m'$

(113e) $\qquad \delta_y\psi^\alpha(t^0) = 0 \quad \text{if} \quad \psi^\alpha(t^0) = 0 \qquad \alpha = 1,\ldots,m'$

(113f) $\qquad \delta_y\psi^\alpha(t) = 0 \qquad t^0 \le t \le t^1 \qquad \alpha = m' + 1,\ldots,m$

(113g) $\qquad \delta_y\theta^\eta(t) = 0$ on a neighborhood of $p^\eta \qquad \eta = 1,\ldots,L'$

(113h) $\qquad \delta_y\theta^\eta(t) = 0 \qquad t^0 \le t \le t^1 \qquad \eta = L' + 1,\ldots,L$

(113i) $\qquad J'_\gamma(a_0,y) = 0 \quad \text{if} \quad \lambda_\gamma \ne 0 \qquad 1 \le \gamma \le p'$

(113j) $\qquad J'_\gamma(a_0,y) \le 0 \quad \text{if} \quad \lambda_\gamma = 0 \qquad \gamma \ne \gamma_k \qquad 1 \le \gamma \le p'$

(113k) $\qquad J'_\rho(a_0,y) = 0 \qquad p' < \rho \le p + 2N$

where(1) $\mu_\alpha(t)$, $\lambda_{p+N+\alpha}$ are multipliers of Theorem 9.1 of [2].

Lastly we list the functions H and G of Theorem 9.1 of [2] as follows:

$$H = p_i(t)f^i - \lambda_0 L_0 - \lambda_\gamma L_\gamma - \mu_\alpha(t)\phi^\alpha - h_\eta(t)\theta^\eta$$

$$G = \lambda_0 g_0 + \lambda_\gamma g_\gamma + \lambda_{p+i}X^{i0} - \lambda_{p+N+\alpha}[\psi^\alpha(t^s,X^s)]_{s=0}^{s=1}$$

$$i = 1,\ldots,N, \qquad \gamma = 1,\ldots,p, \qquad \alpha = 1,\ldots,m, \qquad \eta = 1,\ldots,L$$

The second order theorem to be proven for this problem is then

(1) By a more complicated procedure it is possible to replace (113g) by the less restrictive conditions $\delta_y\theta^\eta(t) \le 0$ on a neighborhood of p^η and $\delta_y\theta^\eta(t) = 0$ if $h_\eta(t) \ne 0 \qquad \eta = 1,\ldots,L'$.

Theorem 11.1 Let a_0 be a normal solution to the problem of section 13. Then the multipliers of Theorem 9.1 of [2], K^α, λ_ρ, $\mu_\alpha(t)$, $h_\eta(t)$, $p_i(t)$

$\rho = 0,1,\ldots,p + N + m$; $\alpha = 1,\ldots,m$; $\eta = 1,\ldots,L$; $i = 1,\ldots,N$

can be selected with $\lambda_0 = 1$. When so selected, the multipliers, λ_0, $\lambda_1,\ldots,\lambda_{p+N}$, $K^1,\ldots,K^m$, $h_1(t),\ldots,h_L(t)$ are unique while the multipliers $\lambda_{p+N+\alpha}$, $\mu_\alpha(t)$, $p_i(t)$ ($\alpha = 1,\ldots,m$; $i = 1,\ldots,N$) are respectively unique up to the added terms c^α, c^α, $c^\alpha\psi^\alpha_{x^i}(t)$ all with the same arbitrary constants c^α ($\alpha = 1,\ldots,m$).

Furthermore with H, G as the functions listed above, then for each variation y in the class Y we have the following inequality

$$(114)\quad 0 \le [(p_i(t^c)X^{ic}_{b^\sigma b^\tau})\Big|_{c=0}^{c=1} + G_{b^\sigma b^\tau} + K^\alpha\psi^\alpha_{x^i x^j}(t^0)X^{i0}_{b^\sigma}X^{j0}_{b^\tau}]z^\sigma z^\tau - \int_{t^0}^{t^1}[H_{x^i x^j}y^i y^j + 2H_{x^i u^h}y^i v^h + H_{u^h u^k}v^h v^k]dt \qquad i,j = 1,\ldots,N \qquad \sigma,\tau = 1,\ldots,r \qquad h,k = 1,\ldots$$

12. Transformation of the Problem

According to [2] the present problem may be written as one of the type of section 8 by introducing the additional control variables

$$(115)\qquad u^{K+\eta} \qquad \eta = 1,\ldots,L'$$

and by defining the functions

$$(116)\qquad \bar\theta^\eta = \theta^\eta + (u^{K+\eta})^2 \qquad \eta = 1,\ldots,L' \qquad \bar\theta^\eta = \theta^\eta \qquad \eta = L' + 1,\ldots,L$$

$$\bar\theta^{L+\alpha} = \phi^{m'+\alpha} \qquad \alpha = 1,\ldots,m - m'$$

We are now concerned with arcs

$$(117)\quad \bar{a}:\quad \bar{x}^i(t) \qquad \bar{u}^k(t) \qquad \bar{b}^\sigma \qquad t^0 \le t \le t^1$$

$$i = 1,\dots,N \qquad k = 1,\dots,K + L' \qquad \sigma = 1,\dots,r$$

and wish to minimize the functional of (107) subject to the constraints (108a) and (108e) together with (108b) through (108d) replaced by

$$(118a)\qquad \psi^\alpha(t,x) \le 0 \qquad \alpha = 1,\dots,m'$$

$$(118b)\qquad \bar{\theta}^\eta(t,x,u) = 0 \qquad \eta = 1,\dots,L + m - m'$$

$$(118c)\qquad \bar{I}_\gamma(\bar{a}) \le 0 \qquad \gamma = 1,\dots,p' \qquad \bar{I}_\gamma(\bar{a}) = 0 \qquad \gamma = p' + 1,\dots,p$$

$$\bar{I}_{p+\alpha}(\bar{a}) = \psi^{m'+\alpha}(t^0,X^0(\bar{b})) = 0 \qquad \alpha = 1,\dots,m - m'$$

(where I_γ has the meaning of section 11) so that we have added $m - m'$ more isoperimetric constraints.

Next, define the arc

$$(119a)\quad \bar{a}_0:\quad \bar{x}^i_0(t) = x^i_0(t) \qquad \bar{u}^h_0(t) = u^h_0(t), \qquad \bar{u}^{k+\eta}_0(t) = \sqrt{-\theta^\eta(t,x_0(t),u_0(t))}$$

$$\bar{b}^\sigma_0 = b^\sigma_0 \qquad t^0 \le t \le t^1 \qquad i = 1,\dots,N \qquad h = 1,\dots,K$$

$$\eta = 1,\dots,L' \qquad \sigma = 1,\dots,r$$

where the unbarred quantities are from the arc a_0 of (109).

As proven in [2], the arc $\bar{a}_0$ will be a solution to the transformed problem if the arc a_0 of (109) is a solution to the original problem.

And corresponding to the condition involving (58b) the transformed problem is such that the matrix

$$\text{(119b)} \quad \begin{bmatrix} \bar{\theta}^{\rho}_{u^k} & \bar{\theta}^{\rho}_{u^{K+\eta}} & 0 \\ \phi^{\alpha}_{u^k} & \phi^{\alpha}_{u^{K+\eta}} & \delta_{\alpha\beta}\psi^{\beta} \end{bmatrix} \quad \begin{array}{ll} \rho = 1,\ldots,L+m-m' & k = 1,\ldots,K \\ \eta = 1,\ldots,L' & \alpha,\beta = 1,\ldots,m' \end{array}$$

has rank $L + m$ on the set $\bar{R}_1$ as defined in section 8.

Thus according to Theorem 6.1 of [2] and section 10 of [2] there are multipliers $\bar{K}^{\alpha}$, $\bar{\lambda}_{\rho}$, $\bar{\mu}_{\alpha}(t)$, $\bar{h}_{\eta}(t)$, $\bar{p}_i(t)$ and functions $\bar{H}$ and $\bar{G}$ of the form

$$\text{(120)} \quad \bar{G} = \bar{\lambda}_0 g_0 + \bar{\lambda}_{\gamma} g_{\gamma} + \bar{\lambda}_{p+\beta}\psi^{m'+\beta}(t^0,X^0) + \bar{\lambda}_{\tilde{p}+i}X^{i0} - \bar{\lambda}_{\tilde{p}+N+\alpha}[\psi^{\alpha}(t^s,X^s)]_{s=0}^{s=1}$$

$$\bar{H} = \bar{p}_i(t)f^i - \bar{\lambda}_0 L_0 - \bar{\lambda}_{\gamma} L_{\gamma} - \bar{\mu}_{\alpha}(t)\phi^{\alpha} - \bar{h}_{\eta}(t)\dot{\bar{\theta}}^{\eta}$$

$i = 1,\ldots,N$; $\eta = 1,\ldots,L + m - m'$; $\alpha = 1,\ldots,m'$, $\rho = 0,1,\ldots,\tilde{p} + N +$

$\tilde{p} = p + m - m'$; $\beta = 1,\ldots,m - m'$; $\gamma = 1,\ldots,p$.

where by section 10 of [2] the relationships between these terms and the terms of Theorem 9.1 of [2] are

$$\lambda_{\gamma} = \bar{\lambda}_{\gamma} \qquad \gamma = 0,1,\ldots,p$$

$$\text{(121)} \quad \lambda_{p+i} = \bar{\lambda}_{\tilde{p}+i} + \bar{\lambda}_{p+\beta}\psi^{m'+\beta}_{x^i}(t^0) \qquad \beta = 1,\ldots,m-m' \qquad i = 1,\ldots,N$$

$$\lambda_{p+N+\alpha} = \bar{\lambda}_{\tilde{p}+N+\alpha} \qquad \alpha = 1,\ldots,m' \qquad \tilde{p} = p + m - m'$$

$$\lambda_{p+N+m'+\beta} = e^{\beta} \qquad \beta = 1,\ldots,m - m'$$

$$\mu_{\alpha}(t) = \bar{\mu}_{\alpha}(t) \qquad \alpha = 1,\ldots,m'$$

$$\mu_{m'+\beta}(t) = \bar{h}_{L+\beta}(t) + e^{\beta} \qquad \beta = 1,\ldots,m - m'$$

$$p_i(t) = \bar{p}_i(t) + e^{\beta}\psi^{m'+\beta}_{x^i}(t) \qquad \beta = 1,\ldots,m - m' \qquad i = 1,\ldots,N$$

(121) $$h_{\eta}(t) = \bar{h}_{\eta}(t) \qquad \eta = 1,\ldots,L$$

$$K^{\alpha} = \bar{K}^{\alpha} \qquad \alpha = 1,\ldots,m'$$

$$K^{m'+\beta} = \bar{\lambda}_{p+\beta} \qquad \beta = 1,\ldots,m - m'$$

$$G = \bar{G} + \bar{\lambda}_{p+\beta}[\psi^{m'+\beta}_{\_i}(t^0)X^{i0} - \psi^{m'+\beta}(t^0,X^0)] - \lambda_{p+N+m'+\beta}\Big[\psi^{m'+\beta}(t^s,X^s)\Big]_{s=0}^{s=1}$$

$$H = \bar{H} + e^{\beta}\psi^{m'+\beta}_{x^i}(t)f^i - e^{\beta}\phi^{m'+\beta} + h_s(t)(u^{k+s})^2$$

$$s = 1,\ldots,L' \qquad \beta = 1,\ldots,m - m'$$

where e^{β} $(\beta = 1,\ldots,m - m')$ are arbitrary constants.

In order now to prove Theorem 11.1, we will first define normality for the transformed problem and then prove that it is normal.

Corresponding to the functionals of section 8, we define the functionals

(122a) $$\bar{J}_{\gamma}(\bar{a}) = J_{\gamma}(\bar{a}) \qquad \gamma = 0,1,\ldots,p$$

$$\bar{J}_{p+\beta}(\bar{a}) = \psi^{m'+\beta}(t^0,X^0) \qquad \beta = 1,\ldots,m - m'$$

(122a) $$\bar{J}_{\tilde{p}+i}(\bar{a}) = J_{p+i}(\bar{a}) \qquad i = 1,\ldots,N$$

$$\bar{J}_{\tilde{p}+N+i}(\bar{a}) = J_{p+N+i}(\bar{a}) \qquad i = 1,\ldots,N \qquad \tilde{p} = p+m-m'$$

where the functionals $J_\rho \quad \rho = 0,1,\ldots,p+2N$ are those of section 11 and where according to the definition of these functionals and of the arcs $\bar{a}$, we may evaluate J_ρ on those arcs and have indicated this by writing $J_\rho(\bar{a})$. We have thus added $m - m'$ more functionals to those of section 11 corresponding to our $m - m'$ additional isoperimetric constraints.

The conditions analagous to (61a) through (61d) for the transformed problem are

(122b) $$\bar{\delta}\dot{\bar{x}}^i = \delta f^i \qquad i = 1,\ldots,N$$

$$\bar{\delta}\psi^\alpha(t^0) = 0 \text{ if } \bar{\mu}_\alpha(t^0) \neq \bar{\lambda}_{\tilde{p}+N+\alpha} \qquad 1 \leq \alpha \leq m'$$

$$\bar{\delta}\psi^\alpha(t) = 0 \text{ on a neighborhood of } S^\alpha \qquad 1 \leq \alpha \leq m'$$

$$\bar{\delta}\bar{\theta}^\eta(t) = 0 \qquad t^0 \leq t \leq t^1 \qquad \eta = 1,\ldots,L + m - m'$$

where S^α is defined as before.

Defining the indices γ_k as in section 8, we see that according to our definition of the transformed problem and of the arc $\bar{a}_0$, then these are the same indices as obtained in section 11. Hence normality for the transformed problem requires the existence of $\bar{\Omega} = \Omega + m - m'$ variations $\bar{\delta}_\tau\bar{a}$:

$$\bar{\delta}_\tau\bar{a}: \qquad \bar{\delta}_\tau\bar{x}^i(t) \qquad \bar{\delta}_\tau\bar{u}^k(t) \qquad \bar{\delta}_\tau\bar{b}^\sigma \qquad t^0 \leq t \leq t^1$$

$$\tau = 1,\ldots,\bar{\Omega} = \Omega + m - m', \qquad i = 1,\ldots,N, \qquad k = 1,\ldots,K + L' \qquad \sigma = 1,$$

(where Ω is the constant of (111)) satisfying (122b) and also providing that the matrix

$$(122c) \qquad (\bar{J}'_\rho(a_0, \bar{\delta}_\tau \bar{a})) \qquad \tau = 1,\ldots,\bar{\Omega} \qquad \rho = 1,\ldots,\tilde{p} + 2N \qquad \rho \neq \gamma_k$$

is non singular (where γ_k are the indices referred to above and where consistent with previous notation $\bar{J}'_\rho(\bar{a}_0, \bar{\delta}_\tau \bar{a})$ denotes the variation of $\bar{J}_\rho$ due to the variation $\bar{\delta}_\tau \bar{a}$ of $\bar{a}_0$)

13. Proof of Normality and the Class $\bar{Y}$ for the Transformed Problem of Section 11.

With the above introduction we now prove

Lemma 13.1 Normality of the problem of section 11 implies the existence of $\bar{\Omega}$ variations $\bar{\delta}_\tau \bar{a}$ ($\tau = 1,\ldots,\bar{\Omega}$) satisfying (122b) and (122c) and hence also implies normality of the transformed problem.

Proof: In order to show the existence of these variations perform the following construction. Using the variations $\delta_\omega a$ defined in (112), define the variations

$$(123) \quad \bar{\delta}_\omega \bar{a}: \qquad \bar{\delta}_\omega \bar{x}^i(t) \qquad \bar{\delta}_\omega \bar{u}^k(t) \qquad \bar{\delta}_\omega \bar{b}^\sigma \qquad t^0 \le t \le t^1 \qquad \omega = 1,\ldots,\Omega$$

(where Ω is the term of (111)) as follows:

$$(124) \qquad \bar{\delta}_\omega \bar{x}^i(t) = \delta_\omega x^i(t) \qquad i = 1,\ldots,N$$

$$\bar{\delta}_\omega \bar{u}^h(t) = \delta_\omega u^h(t) \qquad h = 1,\ldots,k$$

$$\bar{\delta}_\omega \bar{u}^{k+\eta}(t) = \begin{cases} 0 & \text{if } \delta_\omega \theta^\eta(t) = 0 \\ \dfrac{-\delta_\omega \theta^\eta(t)}{2\sqrt{-\theta^\eta(t)}} & \text{otherwise}^{(1)} \end{cases} \qquad \eta = 1,\ldots,L'$$

$$\bar{\delta}_\omega \bar{b}^\sigma = \delta_\omega b^\sigma \qquad \sigma = 1,\ldots,r \qquad \omega = 1,\ldots,\Omega$$

where the unbarred terms are from the variations of (112).

(1) The term $\theta^\eta(t)$ signifies $\theta^\eta(t, x_0(t), u_0(t))$.

With this construction and by the construction of $\bar{a}_0$ together with the properties (112) of $\delta_\omega a$ and the definition of the functions $\bar{\theta}^\eta$ we have

$$(125) \quad \bar{\delta}_\omega \bar{\theta}^\eta = \delta_\omega \theta^\eta + 2\bar{u}_0^{K+\eta}(t)\bar{\delta}_\omega \bar{u}^{K+\eta} = 0 \qquad t^0 \le t \le t^1 \qquad \eta = 1,\dots,L' \ \eta \text{ not summed}$$

$$\bar{\delta}_\omega \bar{\theta}^{L+\alpha-m'} = \bar{\delta}_\omega \phi^\alpha(t) = \phi^\alpha_{x^i}\bar{\delta}_\omega \bar{x}^i + \phi^\alpha_{u^k}\bar{\delta}_\omega \bar{u}^k =$$

$$[\psi^\alpha_{tx^i} + \psi^\alpha_{x^j x^i} f^j + \psi^\alpha_{x^j} f^j_{x^i}]\delta_\omega x^i + \psi^\alpha_{x^j} f^j_{u^k}\delta_\omega u^k =$$

$$[\psi^\alpha_{tx^i} + \psi^\alpha_{x^j x^i} f^j]\delta_\omega x^i + \psi^\alpha_{x^j}[f^j_{x^i}\delta_\omega x^i + f^j_{u^k}\delta_\omega u^k] = \frac{d}{dt}\,\delta_\omega \psi^\alpha(t)$$

$$t^0 \le t \le t^1 \qquad \alpha = m' + 1,\dots,m \qquad k = 1,\dots,K \qquad j,i = 1,\dots,N$$

$$\omega = 1,\dots,\Omega$$

where in (125) the argument is t and we have recognized that the functions ϕ^α $(\alpha = m' + 1,\dots,m)$ do not depend on $u^{K+\eta}$ $(\eta = 1,\dots,L')$ and also have used components of the variations $\bar{\delta}_\omega \bar{a}$ and $\delta_\omega a$ interchangeably when they are the same.

By (125) together with (112) and the relationships indicated in (121), then the variations $\bar{\delta}_\omega \bar{a}$ $(\omega = 1,\dots,\Omega)$ satisfy conditions (122b).

Forming variations of the functions of (122a) with respect to the variations of (124) we see that property (112g) for the variations $\delta_\omega a$ implies that the first $p - \bar{p}$ (1) and last $2N$ rows of the matrix

(1) The term $\bar{p}$ is the number of indices γ_k referred to previously.

$$(126) \qquad \begin{bmatrix} \bar{J}'_{\gamma}(\bar{a}_0,\bar{\delta}_{\omega}\bar{a}) \\ \\ \bar{J}'_{p+\beta}(\bar{a}_0,\bar{\delta}_{\omega}\bar{a}) \\ \\ \bar{J}'_{\tilde{p}+i}(\bar{a}_0,\bar{\delta}_{\omega}\bar{a}) \\ \\ \bar{J}'_{\tilde{p}+N+i}(\bar{a}_0,\bar{\delta}_{\omega}\bar{a}) \end{bmatrix} \begin{array}{l} \gamma = 1,\ldots,p \quad \gamma \neq \gamma_k \quad (k = 1,\ldots,\bar{p}) \\ \\ \beta = 1,\ldots,m - m' \\ \\ \\ \\ i = 1,\ldots,N \quad \tilde{p} = p + m - m' \quad \omega = 1,\ldots,\Omega \end{array}$$

are linearly independent. Normality will be proven when we show the existence of $m - m'$ additional variations

$$\bar{\delta}_{\Omega+\tau}\bar{a} \qquad \tau = 1,\ldots,m - m'$$

which satisfy (122b) and together with the variations $\bar{\delta}_{\omega}\bar{a}$ of (124) satisfy (122c)

The proof of this existence will be done in the following lemma.

Lemma 13.2 Under the conditions existing on the matrix of (126) for the variations of (124), we can construct $m - m'$ additional variations

$$(127) \qquad \bar{\delta}_{\Omega+\tau}\bar{a} \qquad \tau = 1,\ldots,m - m'$$

satisfying the conditions (122b) and also (122c) with the variations $\bar{\delta}_{\omega}\bar{a}$ of (124).

<u>Proof</u>: We shall be concerned with that submatrix of (126) consisting of the variations $\bar{J}'_{p+\beta}(\bar{a}_0,\bar{\delta}_{\omega}\bar{a})$ and $\bar{J}'_{\tilde{p}+N+i}(\bar{a}_0,\bar{\delta}_{\omega}\bar{a})$ ($\beta = 1,\ldots, m - m'$; $i = 1,\ldots,N$) which look like

$$(128)\quad \begin{bmatrix} \bar{J}'_{p+\beta}(\bar{a}_0,\bar{\delta}_\omega\bar{a}) \\ \\ \bar{J}'_{\tilde{p}+N+i}(\bar{a}_0,\bar{\delta}_\omega\bar{a}) \end{bmatrix} = \begin{bmatrix} \psi_x^{m'+\beta}(t^0)\bar{\delta}_\omega X^0 \\ \\ \bar{\delta}_\omega X^{i0} - \bar{\delta}_\omega \bar{x}^i(t^0) \end{bmatrix} \begin{array}{ll} \beta = 1,\ldots,m-m' & i = 1,\ldots,N \\ \\ \omega = 1,\ldots,\Omega & \tilde{p} = p+m-m' \end{array}$$

where[1] i,β vary with the row and ω is the column index. Our task will be to show the existence of $m - m'$ additional variations $\bar{\delta}_{\Omega+\tau}$ $(\tau = 1,\ldots,m-m')$ which satisfy (122b) and also provide that none of the last $m - m'$ columns of the following submatrix of (122c)

$$(129)\quad \begin{bmatrix} \bar{J}'_{p+\beta}(\bar{a}_0,\bar{\delta}_\rho\bar{a}) \\ \\ \bar{J}'_{\tilde{p}+N+i}(\bar{a}_0,\bar{\delta}_\rho\bar{a}) \end{bmatrix} =$$

$$\left[\begin{array}{l|l} \psi_x^{m'+\beta}(t^0)\,\bar{\delta}_1X^0,\ldots,\psi_x^{m'+\beta}(t^0)\bar{\delta}_\Omega X^0, & \psi_x^{m'+\beta}(t^0)\,\bar{\delta}_{\Omega+1}X^0,\ldots,\psi_x^{m'+\beta}(t^0)\bar{\delta}_{\bar{\Omega}}X^0 \\ & \\ \bar{\delta}_1X^{i0} - \bar{\delta}_1\bar{x}^i(t^0),\ldots,\bar{\delta}_\Omega X^{i0} - \bar{\delta}_\Omega\bar{x}^i(t^0), & \bar{\delta}_{\Omega+1}X^{i0} - \bar{\delta}_{\Omega+1}\bar{x}^i(t^0),\ldots,\bar{\delta}_{\bar{\Omega}}X^{i0} - \bar{\delta}_{\bar{\Omega}}\bar{x}^i(\end{array}\right.$$

$$\rho = 1,\ldots,\bar{\Omega}\,,\qquad \beta = 1,\ldots,m-m'\qquad \bar{\Omega} = \Omega + m - m'\qquad \tilde{p} = p+m-m'$$

$$i = 1,\ldots,N$$

can be written in terms of the other $\bar{\Omega} - 1$ columns. This property together with the linear independence of the Ω columns of (126) will then guarantee that the condition involving (122c) is satisfied.

Now according to (109e) the matrix

$$(130)\qquad (\psi_{x^i}^{m'+\beta}(t^0))\qquad \beta = 1,\ldots,m-m'$$

(1) Note that for convenience, we have omitted the index j in $\psi_x^{m'+\beta}(t^0)\bar{\delta}_\omega X^0$ this term meaning $\psi_{x^j}^{m'+\beta}(t^0)\bar{\delta}_\omega X^{j0}$. This notation will be used again.

has rank $m - m'$ and according to construction of the variation $\bar{\delta}_{\omega}\bar{a}$ of (124) and the properties of the variations of (112) then we have

$$\psi^{m'+\beta}_{x^i}(t^0)\bar{\delta}_{\omega}\bar{x}^i(t^0) = 0 \qquad \omega = 1,\ldots,\Omega \qquad \beta = 1,\ldots,m - m' \tag{131}$$

Furthermore, by the condition on (126), the following submatrix formed by its last N rows,

$$\begin{bmatrix} \bar{\delta}_1 X^{1,0} - \bar{\delta}_1\bar{x}^1(t^0),\ldots,\bar{\delta}_{\Omega}X^{1,0} - \bar{\delta}_{\Omega}\bar{x}^1(t^0) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \bar{\delta}_1 X^{N,0} - \bar{\delta}_1\bar{x}^N(t^0),\ldots,\bar{\delta}_{\Omega}X^{N,0} - \bar{\delta}_{\Omega}\bar{x}^N(t^0) \end{bmatrix} \tag{132}$$

has rank N . Next, with

$$\bar{\delta}_{\omega}X^0, \qquad \bar{\delta}_{\omega}\bar{x}(t^0) \qquad \omega = 1,\ldots,\Omega \tag{133}$$

as the column vectors whose differences form the columns of (132), define the space V as the space of vectors spanned by $\bar{\delta}_{\omega}X^0$ $(\omega = 1,\ldots,\Omega)$ and the space M as the space of vectors which are perpendicular to the $m - m'$ gradients $\psi^{m'+\beta}_{x^i}(t^0)$ $(\beta = 1,\ldots,m - m')$. Thus for example M contains the vectors $\bar{\delta}_{\omega}\bar{x}(t^0)$ $(\omega = 1,\ldots\Omega)$

Then any N component vector Z can be written as the difference

$$Z = \gamma_{\omega}(\bar{\delta}_{\omega}X^0 - \bar{\delta}_{\omega}\bar{x}^i(t^0)) \qquad \omega = 1,\ldots,\Omega \tag{134}$$

for suitable γ_{ω} .

Now since the matrix (130) has rank $m - m'$ then for any $m - m'$ linearly independent $m - m'$ component vectors $T_1, \ldots, T_{m - m'}$ we can find N component vectors $\bar{Z}_1, \ldots, \bar{Z}_{m - m'}$ such that

$$(135) \qquad (\psi_x^{m'+\beta}(t^0))\bar{Z}_j = T_j \qquad j = 1, \ldots, m - m'$$

By representing $\bar{Z}_j$ as in (134) and by (131) we see that this means

$$(136) \quad T_j = (\psi_x^{m'+\beta}(t^0))\bar{Z}_j = (\psi_x^{m'+\beta}(t^0))\gamma_{j\omega}(\bar{\delta}_\omega X^0 - \bar{\delta}_\omega \bar{x}(t^0)) = (\psi_x^{m'+\beta}(t^0))\gamma_{j\omega}\bar{\delta}_\omega X^0$$

$$j = 1, \ldots, m - m'$$

so that by proper selection of $\gamma_{j\omega}$ we see that there are $m - m'$ linearly independent vectors of the form

$$(137) \qquad (\psi_x^{m'+\beta}(t^0))\gamma_{j\omega}\bar{\delta}_\omega X^0 \qquad j = 1, \ldots, m - m' \qquad \omega = 1, \ldots, \Omega$$

where the vectors $\bar{\delta}_\omega X^0$ are due to the variations of (124).

Next, let M denote the matrix of (109h)

$$(138) \qquad M = \begin{bmatrix} \psi^{\alpha_j}_{x^i}(t^0) \\ \\ \psi^{\gamma_s}_{x^i}(t^0) \\ \\ \psi^{m'+\beta}_{x^i}(t^0) \end{bmatrix} \qquad \begin{array}{ll} 1 \le \gamma_s, \alpha_j \le m' & s = 1, \ldots, \bar{\bar{m}} \\ \\ j = 1, \ldots, \hat{n} & \beta = 1, \ldots, m - m' \\ \\ i = 1, \ldots, N & \end{array}$$

By our assumptions, the last $m - m'$ rows of (M) are linearly independent and are independent of the other rows. Then with the vector $E = (e_1,\ldots,e_{\bar{\bar{m}} + \hat{n} + m - m'})$ having its first $\bar{\bar{m}} + \hat{n}$ components zero and last $m - m'$ components arbitrary, we may solve the system

$$\text{(139)} \qquad (M) \begin{bmatrix} d^1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ d^N \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ e_{\bar{\bar{m}} + \hat{n} + 1} \\ \cdot \\ \cdot \\ \cdot \\ e_{\bar{\bar{m}} + \hat{n} + m - m'} \end{bmatrix}$$

for the vector $D = (d^1,\ldots,d^N)$. Furthermore, for linearly independent vectors E of the form in (139) then the associated D vectors will also be linearly independent. Thus choosing $m - m'$ such linearly independent vectors $E_1,\ldots,E_{m - m'}$ then the vectors $D_1,\ldots,D_{m - m'}$ are linearly independent.

We note here according to the last $m - m'$ rows of (139) that the equations $(\psi_x^{m'+\beta}(t^0))D = 0$, $(\beta = 1,\ldots,m - m')$ are not all satisfied by any of the vectors $D_1,\ldots,D_{m - m'}$ so that these vectors are not in the space $\mathcal{M}$.

We have thus shown that according to (137) if we select the $\bar{\delta}\bar{b}$ part of our required variations $\bar{\delta}_{\Omega+j}$ $(j = 1,\ldots,m - m')$ as

$$\bar{\delta}_{\Omega+j}\bar{b}^{\sigma} = \gamma_{j\omega}\bar{\delta}_{\omega}\bar{b}^{\sigma} \qquad j = 1,\ldots,m - m' \qquad \omega = 1,\ldots,\Omega \tag{140}$$

with $\gamma_{j\omega}$ as specified in (137), then the vectors

$$(\psi^{m'+\beta}_{x^i}(t^0))X^{i0}_{b^\sigma}\bar{\delta}_{\Omega+j}\bar{b}^{\sigma} = (\psi^{m'+\beta}_{x^i}(t^0))X^{i0}_{b^\sigma}\gamma_{j\omega}\bar{\delta}_{\omega}\bar{b}^{\sigma} = (\psi^{m'+\beta}_{x^i}(t^0))\gamma_{j\omega}\bar{\delta}_{\omega}X^{i} \tag{141}$$

$$j = 1,\ldots,m - m'$$

are linearly independent and that furthermore by (139) we can find $m - m'$ linearly independent vectors D_i not in the space M. Next consider the matrix

$$\left[\begin{array}{c|c} \bar{\delta}_{\omega}X^0 - \bar{\delta}_{\omega}\bar{x}(t^0) & \bar{\delta}_{\Omega+j}X^0 - D_j \\ \psi^{m'+\beta}_{x}(t^0)\bar{\delta}_{\omega}X^0 & \psi^{m'+\beta}_{x}(t^0)\bar{\delta}_{\Omega+j}X^0 \end{array}\right] \qquad \begin{array}{l}\omega = 1,\ldots,\Omega \\ j,\beta = 1,\ldots,m - \end{array} \tag{142}$$

where: i) $\bar{\delta}_{\omega}\bar{x}(t^0)$, $\bar{\delta}_{\omega}X^0$ come from the variations of (124) and ii) $\bar{\delta}_{\Omega+j}X^0$ and D_j $(j = 1,\ldots,m - m')$ are from the above constructions. We show now that none of the last $m - m'$ columns of (142) can be written as a linear combination of all the other $\bar{\Omega} - 1 = \Omega + m - m' - 1$ columns.

Assume that this is not the case so that say for $\bar{j}$, $(\Omega + 1 \leqq \bar{j} \leqq \bar{\Omega})$ we have for suitable constants α

$$\alpha_{\omega}[\bar{\delta}_{\omega}X^0 - \bar{\delta}_{\omega}\bar{x}(t^0)] + \alpha_{\Omega+j}[\bar{\delta}_{\Omega+j}X^0 - D_j] = \bar{\delta}_{\Omega+\bar{j}}X^0 - D_{\bar{j}} \tag{143a}$$

$$\alpha_{\omega}(\psi^{m'+\beta}_{x}(t^0))\bar{\delta}_{\omega}X^0 + \alpha_{\Omega+j}(\psi^{m'+\beta}_{x}(t^0))\bar{\delta}_{\Omega+j}X^0 = (\psi^{m'+\beta}_{x}(t^0))\bar{\delta}_{\Omega+\bar{j}}X^0 \tag{143b}$$

$$\omega = 1,\ldots,\Omega \qquad \beta,j = 1,\ldots,m - m' \qquad j \neq \bar{j}$$

According to (143b) we have

$$(144a)\qquad (\psi_x^{m'+\beta}(t^0))\ [\alpha_\omega \bar{\delta}_\omega X^0 - \alpha_{\Omega+j}\bar{\delta}_{\Omega+j}X^0 - \bar{\delta}_{\Omega+\bar{j}}X^0] = 0$$

$$\omega = 1,\ldots,\Omega; \quad \beta, j = 1,\ldots,m - m'; \qquad j \neq \bar{j}$$

so that by definition of the space M, then the vector in brackets is in M.

But also from (143a) we have that

$$(144b)\qquad \alpha_{\Omega+j}D_j - D_{\bar{j}} = [\alpha_\omega \bar{\delta}_\omega X^0 + \alpha_{\Omega+j}\bar{\delta}_{\Omega+j}X^0 - \bar{\delta}_{\Omega+\bar{j}}X^0] - \alpha_\omega \bar{\delta}_\omega \bar{x}(t^0)$$

$$j = 1,\ldots,m - m', \qquad j \neq \bar{j}, \qquad \omega = 1,\ldots,\Omega$$

which according to the above remarks implies that a linear combination of the vectors D_j is in the space M. This means that with (M), E_j as the quantities of (139) we have

$$(145)\qquad M(\alpha_{\Omega+j}D_j - D_{\bar{j}}) = (\alpha_{\Omega+j}E_j - E_{\bar{j}}) = 0 \qquad j = 1,\ldots,m - m' \qquad j \neq \bar{j}$$

which is impossible according to our construction. Thus our statement is true and our proof of the lemma will be complete if we can show that we can define $m - m'$ variations $\bar{\delta}_{\Omega+j}\bar{a}$ which satisfy the conditions (122b) with $\bar{\delta}_{\Omega+j}\bar{x}(t^0) = D_j$ (where the vectors D_j are defined below (139)) and with $\bar{\delta}_{\Omega+j}\bar{b}$ defined in (140).

In order to prove the required statement we perform a contruction similar to that of section 13 of [1] and define the constant

$$(146)\qquad \tilde{K} = K + L' + m'$$

and the class $\mathcal{W}$ of all K dimensioned arcs W

$$(147) \quad W: \qquad w^{\Gamma}(t) \qquad t^0 \le t \le t^1 \qquad \Gamma = 1,\ldots,\tilde{K}$$

which have components w^{Γ}, $(\Gamma = m + 1,\ldots,\tilde{K})$ and the derivatives $\dot{w}^{\alpha}$ of the components w^{α} $(\alpha = 1,\ldots,m)$ piecewise continuous and in addition satisfy the conditions

$$(148a) \qquad w^{\alpha}(t) = 0 \text{ on a neighborhood of } S^{\alpha} \qquad \alpha = 1,\ldots,m'$$

$$(148b) \qquad w^{\alpha}(t^0) = 0 \text{ if } \bar{\mu}_{\alpha}(t^0) \ne \bar{\lambda}_{\tilde{p}+N+\alpha} \qquad 1 \le \alpha \le m' \quad \tilde{p} = p + m - m'$$

$$(148c) \qquad w^{m'+\beta}(t^0) \ne 0 \text{ for some } \beta \qquad 1 \le \beta \le m - m'$$

$$(148d) \qquad \dot{w}^{m'+\beta}(t) = 0 \qquad t^0 \le t \le t^1 \qquad \beta = 1,\ldots,m - m'$$

$$(148e) \qquad w^{m+\eta}(t) = 0 \qquad t^0 \le t \le t^1 \qquad \eta = 1,\ldots,L$$

where S^{α} has the usual definition and $\bar{\mu}_{\alpha}(t)$, $\bar{\lambda}_{\tilde{p}+N+\alpha}$ are the terms listed above (1[illegible])

If $\bar{\delta}\bar{a}$ is a variation of $\bar{a}_0$ which satisfies the conditions (122b) then with

$$(149) \qquad w^{\alpha}(t) = \delta\psi^{\alpha}(t), \quad \dot{w}^{m'+\beta}(t) = \delta\bar{\theta}^{L+\beta}(t), \quad w^{m+\eta}(t) = \delta\bar{\theta}^{\eta},$$

$$\alpha = 1,\ldots,m', \qquad \beta = 1,\ldots,m - m', \qquad \eta = 1,\ldots,L$$

the arc W with first m + L components as specified above will satisfy all the conditions (148) except for (148c) and may also satisfy that condition although that is not required. We shall prove the converse of this statement, namely that given an arc W satisfying (148), then one can find a variation $\bar{\delta}\bar{a}$ which is related to W through (149) and satisfies (122b). We shall furthermore construct this variation such that we use the vectors D and $\alpha_{j\omega}\bar{\delta}_{\omega}\bar{b}$ as discussed below (145).

In order to do this, we note firstly that if π is any N component vector, satisfying

(150a)
$$\psi^{\alpha}_{x^i}(t^0)\pi^i = 0 \quad \text{if} \quad \psi^{\alpha}(t^0) = 0 \qquad 1 \leq \alpha \leq m'$$

$$\psi^{\alpha}_{x^i}(t^0)\pi^i = 0 \quad \text{if} \quad \bar{\mu}_{\alpha}(t^0) \neq \bar{\lambda}_{p+N+\alpha} \qquad 1 \leq \alpha \leq m'$$

$$\psi^{m'+\beta}_{x^i}(t^0)\pi^i \neq 0 \quad \text{for some} \quad \beta \qquad 1 \leq \beta \leq m - m'$$

then there is an arc W in the class $\mathcal{W}$ which satisfies[1]

(150b)
$$w^{\alpha}(t^0) = \psi^{\alpha}_{x^i}(t^0)\pi^i \qquad \alpha = 1,\ldots,m$$

The vectors D_j $j = 1,\ldots,m - m'$ of (139) satisfy (150a) and we now prove the last item necessary for proving Lemma 13.2 and hence Lemma 13.1.

Lemma 13.3 Given a vector D as described below (139) then one can find a variation $\bar{\delta}\bar{a}$ satisfying the conditions (122b) with

$$\bar{\delta}\bar{x}(t^0) = D \qquad \bar{\delta}\bar{b}^{\sigma} = \alpha_{\bar{j}\omega}\,\bar{\delta}_{\omega}\bar{b}^{\sigma}$$

for any index $\bar{j}$ ($1 \leq \bar{j} \leq m - m'$) where this last vector on the right is the vector for $\bar{j}$ from (140).

<u>Proof</u>: Let W be an arc in $\mathcal{W}$ satisfying (150a) for our vector D. According to the conditions on the matrix (119b) we have that

(151)
$$\begin{bmatrix} \bar{\theta}^{\eta}_{u^k}(t) & 0 \\ \phi^{\alpha}_{u^k}(t) & \delta_{\alpha\beta}\psi^{\beta}(t) \end{bmatrix} \qquad \begin{matrix} \eta = 1,\ldots,L + m - m' \qquad k = 1,\ldots,K + L' \\ \alpha,\beta = 1,\ldots,m' \qquad \beta \text{ not summed} \end{matrix}$$

(1) As an example we can set $\dot{w}^{\Gamma}(t) \equiv 0 \qquad t^0 \leq t \leq t^1 \qquad \Gamma = 1,\ldots,\tilde{K}$

has rank $m + L$ on $[t^0, t^1]$. This means that for any point $\bar{t}$ $(t^0 \leq \bar{t} \leq t^1)$ with $\bar{\alpha}_j$ $(j = 1, \ldots, \bar{m})$ as the indices for which

(152) $$\psi^{\alpha}(\bar{t}) = 0 \qquad 1 \leq \alpha \leq m'$$

then also the rows of the matrix

(153) $$\begin{bmatrix} \theta^{\eta}_{u^k}(\bar{t}) & 0 \\ \\ \phi^{\bar{\alpha}_j}_{u^k}(\bar{t}) & 0 \end{bmatrix} \qquad \begin{array}{ll} j = 1, \ldots, \bar{m} & \eta = 1, \ldots, L + m - m' \\ k = 1, \ldots, K + L' & \end{array}$$

are linearly independent. Now by continuity considerations of the functions involved we have that this independence exists on some neighborhood of $\bar{t}$ and a similar result holds for each point t in $[t^0, t^1]$. As a result, we can modify the functions $\psi^{\alpha}(t)$ $(\alpha = 1, \ldots, m)$ to functions $R^{\alpha}(t)$ of class C^1 which vanish on neighborhoods of S^{α} (the sets where $\psi^{\alpha}(t)$ vanishes) and such that the matrix

(154) $$\begin{bmatrix} \bar{\theta}^{\eta}_{u^k}(t) & 0 \\ \\ \phi^{\alpha}_{u^k}(t) & \delta_{\alpha\beta} R^{\beta}(t) \end{bmatrix} \qquad \begin{array}{l} \eta = 1, \ldots, L + m - m' \\ k = 1, \ldots, K + L' \\ \alpha, \beta = 1, \ldots, m' \end{array}$$

has rank $L + m$ on $[t^0, t^1]$.

Next, introduce the variables

(155) $$Z^{\alpha},\ C^k,\ \mathcal{D}^i,\ V^{\alpha} \quad \alpha = 1, \ldots, m', \quad k = 1, \ldots, K + L', \quad i = 1, \ldots, N$$

Also introduce the functions

$$(156)\qquad \bar{\Delta}^{\alpha}(t,\mathcal{D},Z) = \psi^{\alpha}_{x^i}(t)\mathcal{D}^i + Z^{\alpha}R^{\alpha}(t) \qquad \alpha = 1,\dots,m'$$

$$\dot{\bar{\Delta}}(t,\mathcal{D},Z,C,V) = \phi^{\alpha}_{x^i}(t)\mathcal{D}^i + \phi^{\alpha}_{u^k}(t)C^k + V^{\alpha}R^{\alpha}(t) + Z^{\alpha}\dot{R}^{\alpha}(t) \qquad \alpha = 1,\dots,m'$$

$$\bar{\Delta}^{m'+\beta}(t,\mathcal{D},Z) = \psi^{m'+\beta}_{x^i}(t)\mathcal{D}^i \qquad \beta = 1,\dots, m - m'$$

$$\dot{\bar{\Delta}}^{m'+\beta}(t,\mathcal{D},C) = \bar{\theta}^{L+\beta}_{x^i}(t)\mathcal{D}^i + \bar{\theta}^{L+\beta}_{u^k}(t)C^k \qquad \beta = 1,\dots,m - m'$$

$$\bar{\Delta}^{m+\eta}(t,\mathcal{D},C) = \bar{\theta}^{\eta}_{x^i}(t)\mathcal{D}^i + \bar{\theta}^{\eta}_{u^k}(t)C^k \qquad \eta = 1,\dots,L$$

where α is not summed in (156). Then by the definition of ϕ^{α}, $\bar{\theta}^{L+\beta}$, we see that for functions $\mathcal{D}^i(t)$, $Z^{\alpha}(t)$, $C^k(t)$, $V^{\beta}(t)$ which satisfy

$$(157a)\qquad \dot{\mathcal{D}}^i = f^i_{x^j}\mathcal{D}^j + f^i_{u^k}C^k \qquad \dot{Z}^{\beta} = V^{\beta}$$

$$\beta = 1,\dots,m', \quad i,j = 1,\dots,N, \quad k = 1,\dots,K+L'$$

(where the unlisted argument is t) then

$$(157b)\qquad \dot{\bar{\Delta}}^{\alpha} = \frac{d}{dt}\Delta^{\alpha} \qquad \alpha = 1,\dots,m$$

Hence, the use of the dot superscript which denotes time differentiation in defining these terms is justified.

Now according to the above construction we see that

$$(158)\qquad \begin{bmatrix} \dot{\bar{\Delta}}^{\rho}_{C^k} & \dot{\bar{\Delta}}^{\rho}_{V^{\tau}} \\ \bar{\Delta}^{m+\eta}_{C^k} & \bar{\Delta}^{m+\eta}_{V^{\tau}} \end{bmatrix} = \begin{bmatrix} \phi^{\alpha}_{u^k} & \delta_{\alpha\tau}R^{\tau} \\ \bar{\theta}^{L+\beta}_{u^k} & 0 \\ \bar{\theta}^{\eta}_{u^k} & 0 \end{bmatrix} \qquad \begin{array}{ll} \rho = 1,\dots,m & \alpha,\tau = 1,\dots,m' \\ \eta = 1,\dots,L & k = 1,\dots,K + L' \\ \beta = 1,\dots,m - m' & \end{array}$$

Then by the statement involving (154), these matrices have rank $L + m$ on $[t^0,t^1]$.

Now the functions in the left matrix of (158) are continuous and by the above statements, we can define $\tilde{K} - (L + m)$ $(\tilde{K} = K + L' + m')$ additional functions

(159a) $$\bar{\Delta}^{\rho}(t,C,V) \qquad \rho = m + L + 1,\ldots,\tilde{K}$$

such that the matrix

(159b) $$\begin{bmatrix} \dot{\bar{\Delta}}^{\rho}_{C^k} & \dot{\bar{\Delta}}^{\rho}_{V^\beta} \\ \bar{\Delta}^{\tau}_{C^k} & \bar{\Delta}^{\tau}_{V^\beta} \end{bmatrix} \qquad \begin{matrix} \rho = 1,\ldots,m & k = 1,\ldots,K + L' \\ \beta = 1,\ldots,m' & \tau = m + 1,\ldots,\tilde{K} \end{matrix}$$

is non singular on $[t^0,t^1]$ and all rows are continuous functions in time.

Next, consider the system of equations

(160) $$\dot{\bar{\Delta}}^{\rho}_{C^k}(t)C^k + \dot{\bar{\Delta}}^{\rho}_{V^\alpha}(t)V^\alpha = \dot{w}^{\rho}(t) - \dot{\bar{\Delta}}^{\rho}_{\mathcal{D}^i}(t)\mathcal{D}^i - \dot{\bar{\Delta}}^{\rho}_{Z^\alpha}(t)Z^\alpha$$

$$\bar{\Delta}^{\tau}_{C^k}(t)C^k + \bar{\Delta}^{\tau}_{V^\alpha}(t)V^\alpha = w^{\tau}(t) - \bar{\Delta}^{\tau}_{\mathcal{D}^i}(t)\mathcal{D}^i - \bar{\Delta}^{\tau}_{Z^\alpha}(t)Z^\alpha \qquad t^0 \le t \le t^1$$

$$\alpha = 1,\ldots,m'; \qquad \rho = 1,\ldots,m; \qquad k = 1,\ldots,K + L'$$

$$i = 1,\ldots,N; \qquad \tau = m + 1,\ldots,\tilde{K}$$

where $\dot{w}^{\rho}$, $w^{m+\eta}$ refer to the arc W of this lemma.

By the form of the constructed functions, the functional determinant of these equations with respect to C^k, V^α is the matrix of (158) and by the non-singularity of that matrix we may solve (160) for $C^k(t,\mathcal{D},Z)$, $V^\alpha(t,\mathcal{D},Z)$ and these solutions will be piecewise continuous in t and linear in $\mathcal{D}$, Z .

Next, consider the system of differential equations

$$\text{(161a)} \quad \dot{\mathcal{D}}^i = f^i_{x^j}(t)\mathcal{D}^j + f^i_{u^k}(t)C^k(t,\mathcal{D},Z) \qquad j,i = 1,\ldots,N \qquad k = 1,\ldots,K + L'$$

$$\dot{Z} = V^\alpha(t,\mathcal{D},Z) \qquad \alpha = 1,\ldots,m' \qquad t^0 \le t \le t^1$$

with

$$\text{(161b)} \quad \mathcal{D}(t^0) = D$$

$$Z(t^0) = 0$$

where D is the vector of this lemma. By the properties of f^i, $C^k(t,\mathcal{D},Z)$, $V^\alpha(t,\mathcal{D},Z)$, this system has solution $\mathcal{D}(t)$, $Z(t)$ which yields the functions $C(t) = C(t,\mathcal{D}(t),Z(t))$ and $V(t) = V(t,\mathcal{D}(t),Z(t))$ for $t^0 \le t \le t^1$.

By (161),(160) and the definition of the functions in (156) we see that with these quantities, then

$$\text{(162)} \quad \bar{\Delta}^\alpha(t^0) = \psi^\alpha_{x^i}(t^0)\mathcal{D}^i(t^0) + Z^\alpha(t^0)R^\alpha(t^0) = \psi^\alpha_{x^i}(t^0)\mathcal{D}^i = W^\alpha(t^0) \quad \alpha = 1,\ldots,m'$$

$$\bar{\Delta}^{m'+\beta}(t^0) = \psi^{m'+\beta}_{x^i}(t^0)\mathcal{D}^i(t^0) = \psi^{m'+\beta}_{x^i}(t^0)\mathcal{D}^i = W^{m'+\beta}(t^0) \qquad \beta = 1,\ldots,m - m'$$

$$\dot{\bar{\Delta}}^\alpha(t) = \phi^\alpha_{x^i}(t)\mathcal{D}^i(t) + \phi^\alpha_{u^k}(t)C^k(t) + V^\alpha(t)R^\alpha(t) + Z^\alpha(t)\dot{R}^\alpha(t) = \dot{W}^\alpha(t)$$

$$\alpha = 1,\ldots,m' \qquad t^0 \le t \le t^1$$

$$\dot{\bar{\Delta}}^{m'+\beta}(t) = \bar{\theta}^{L+\beta}_{x^i}(t)\mathcal{D}^i(t) + \bar{\theta}^{L+\beta}_{u^k}(t)C^k(t) = \dot{W}^{m'+\beta}(t) \qquad t^0 \le t \le t^1$$

$$\beta = 1,\ldots,m - m'$$

$$\bar{\Delta}^{m+\eta}(t) = \bar{\theta}^\eta_{x^i}(t)\mathcal{D}^i(t) + \bar{\theta}^\eta_{u^k}(t)C^k(t) = W^{m+\eta}(t) \qquad t^0 \le t \le t^1 \quad \eta = 1,\ldots,L$$

where α is not summed in (162). Then by (162) together with (161) and (157) we have

(163) $$\bar{\Delta}^{\rho}(t) = w^{\rho}(t) \qquad \rho = 1,\dots,L + m^{(1)}$$

Next, define the variation

(164) $$\bar{\bar{\delta}}\bar{a}: \qquad \bar{\bar{\delta}}\bar{x}(t) = \mathcal{D}(t) \qquad \bar{\bar{\delta}}\bar{u}(t) = C(t) \qquad \bar{\bar{\delta}}\bar{b} = \alpha_{\bar{j}\omega}\bar{\delta}_{\omega}\bar{b} \qquad t^0 \le t \le t^1$$

where $\alpha_{\bar{j}\omega}\bar{\delta}_{\omega}\bar{b}$ are the terms of this lemma. Then according to (162) together with the definition of the functions $\bar{\Delta}$, we have

(165)
$$\bar{\delta}\psi^{\alpha}(t) \equiv \psi^{\alpha}_{x^i}(t)\bar{\bar{\delta}}x^i(t) = \bar{\Delta}^{\alpha}(t) - Z^{\alpha}(t)R^{\alpha}(t) \qquad t^0 \le t \le t^1 \qquad \alpha = 1,\dots,m', \alpha \text{ not su}$$
$$\bar{\delta}\psi^{m'+\beta}(t^0) \equiv \psi^{m'+\beta}_{x^i}(t^0)\bar{\bar{\delta}}x^i(t^0) = \bar{\Delta}^{m'+\beta}(t^0) \qquad \beta = 1,\dots,m - m'$$
$$\bar{\bar{\delta}}\theta^{L+\beta}(t) = \bar{\theta}^{L+\beta}_{x^i}(t)\bar{\bar{\delta}}x^i(t) + \bar{\theta}^{L+\beta}_{u^k}(t)\bar{\bar{\delta}}u^k(t) = \dot{\bar{\Delta}}^{m'+\beta}(t) \qquad \beta = 1,\dots,m - m' \qquad t^0 \le t$$
$$\bar{\bar{\delta}}\theta^{\eta}(t) = \bar{\theta}^{\eta}_{x^i}(t)\bar{\bar{\delta}}x^i(t) + \bar{\theta}^{\eta}_{u^k}(t)\bar{\bar{\delta}}u^k(t) = \bar{\Delta}^{m+\eta}(t) \qquad \eta = 1,\dots,L \qquad t^0 \le t \le t^1$$

so that by (163), (161b) and (162) we see that

(166)
$$\bar{\delta}\psi^{\alpha}(t) = W^{\alpha}(t) \quad \text{when} \quad R^{\alpha}(t) = 0 \qquad \alpha = 1,\dots,m' \qquad t^0 \le t \le t^1$$
$$\bar{\delta}\psi^{\alpha}(t^0) = W^{\alpha}(t^0) \qquad \alpha = 1,\dots,m$$
$$\bar{\bar{\delta}}\theta^{\eta}(t) = W^{m+\eta}(t) \qquad t^0 \le t \le t^1 \qquad \eta = 1,\dots,L$$
$$\bar{\bar{\delta}}\theta^{L+\beta}(t) = \dot{W}^{m'+\beta}(t) \qquad t^0 \le t \le t^1 \qquad \beta = 1,\dots,m - m'$$

(1) In fact for $\rho = 1,\dots,\tilde{K}$ this is true but the present statement suffices our requirements.

Thus according to (166) together with (161), the definition of the arc W, and the construction of the functions $R^\alpha(t)$, we have that the variation $\bar{\bar{\delta}}\bar{a}$ of (164) satisfies conditions (122b).

Furthermore by construction, we have our variation satisfying the conditions on $\bar{\bar{\delta}}\bar{x}(t^0)$ and $\bar{\bar{\delta}}\bar{b}$ as specified in the lemma. Thus Lemma 13.3 is proven and by applying this lemma to create the $m - m'$ required variations, also Lemmas 13.2 and 13.1 are proven and the transformed problem is normal.

14. <u>The Class $\bar{Y}$ for the Transformed Problem of Section 11 and the Proof of Theorem 11.1.</u>

Define the class $\bar{Y}$ of variations

$$(167)\quad \bar{y}:\ \bar{y}^i(t)\quad \bar{v}^k(t)\quad \bar{z}^\sigma\quad t^0 \le t \le t^1\quad i = 1,\dots,N;\quad k = 1,\dots,K + L'\quad \sigma = 1,\dots,r$$

satisfying:

$$\dot{\bar{y}}^i = \bar{\delta}_{\bar{y}} f^i \qquad i = 1,\dots,N$$

$$\bar{\delta}_{\bar{y}}\psi^\alpha(t) \le 0 \text{ on a neighborhood of } S^\alpha \qquad \alpha = 1,\dots,m'$$

$$\bar{\delta}_{\bar{y}}\psi^\alpha(t^c) = 0 \text{ if } \bar{\mu}_\alpha(t^c) \ne \bar{\lambda}_{\tilde{p}+N+\alpha} \qquad c = 0,1 \qquad \alpha = 1,\dots,m'$$

$$\bar{\delta}_{\bar{y}}\psi^\alpha(\bar{t}) = 0 \text{ if } \bar{\mu}_\alpha(t) \text{ is not constant on a neighborhood of } \bar{t} \qquad \alpha = 1,\dots,m'$$

$$(168)\qquad \bar{\delta}_{\bar{y}}\psi^\alpha(t^0) = 0 \text{ if } \psi^\alpha(t^0) = 0 \qquad \alpha = 1,\dots,m'$$

$$\bar{\delta}_{\bar{y}}\bar{\theta}^\eta(t) = 0 \qquad t^0 \le t \le t^1 \qquad \eta = 1,\dots,L + m - m'$$

$$\bar{J}'_\gamma(\bar{a}_0,\bar{y}) = 0 \text{ if } \bar{\lambda}_\gamma \ne 0 \qquad 1 \le \gamma \le p'$$

$$\bar{J}'_\gamma(\bar{a}_0,\bar{y}) \le 0 \text{ if } \bar{\lambda}_\gamma = 0 \qquad \gamma \ne \gamma_k \qquad 1 \le \gamma \le p'$$

$$\bar{J}'_\rho(\bar{a}_0,\bar{y}) = 0 \qquad p' < \rho \le \tilde{p} + 2N \qquad \tilde{p} = p + m - m'$$

where: $\bar{\mu}_\alpha(t), \bar{\lambda}_{\tilde{p}+N+\alpha}$ are multipliers of Theorem 9.1 of [2].

The following lemma is proven in an exactly analagous manner to that used in obtaining the variations $\bar{\delta}_\omega \bar{a}$ of (124) in Lemma 13.1.

<u>Lemma 14.1</u> Given a variation y in the class Y then one can construct a variation $\bar{y}$ in the class $\bar{Y}$.

Proof: Given a variation y in Y

$$(169)\quad y: \qquad y^i(t) \qquad v^k(t) \qquad z^\sigma \qquad t^0 \le t \le t^1$$

$$i = 1,\ldots,N \qquad k = 1,\ldots,K \qquad \sigma = 1,\ldots,r$$

then analagous to the construction of (124) construct the variation $\bar{y}$ with

$$(170)\qquad \bar{y}^i(t) = y^i(t) \qquad i = 1,\ldots,N$$

$$\bar{v}^h(t) = v^h(t) \qquad h = 1,\ldots,K$$

$$\bar{v}^{K+\eta}(t) = \begin{cases} 0 & \text{if } \delta_y\theta^\eta(t) = 0 \\ \dfrac{-\delta_y\theta^\eta(t)}{2\sqrt{-\theta^\eta(t)}} & \text{otherwise} \qquad \eta = 1,\ldots,L' \end{cases}$$

$$\bar{z}^\sigma = z^\sigma \qquad \sigma = 1,\ldots,r$$

The remaining steps in the proof of this lemma are analagous to those used in proving that the variations of (124) satisfied (122b) and will not be repeated here. The only item of difference is in establishing that

$$\bar{J}'_{p+\beta}(\bar{a}_0, \bar{y}) = 0 \qquad \beta = 1,\ldots,m - m'$$

However, this follows immediately from the relation

$$\bar{J}'_{p+\beta}(\bar{a}_0,\bar{y}) = \psi^{m'+\beta}_{x^i}(t^0)\bar{\delta}_{\bar{y}}X^{i0} = \psi^{m'+\beta}_{x^i}(t^0)\delta_{y}X^{i0} = \psi^{m'+\beta}_{x^i}(t^0)y^i(t^0) = 0 \qquad \beta = 1,\dots,m - m'$$

where the last two equalities follow from properties (113f) and (113k) for the variation y.

We next prove:

Lemma 14.2 Theorem 8.1 implies Theorem 11.1.

<u>Proof</u>: Applying Theorem 8.1 to the transformed problem and using the relations (121) between terms of Theorems 8.1 and 11.1, we see that the first two statements of Theorem 8.1 imply the first two statements of Theorem 11.1.

In order now to establish the inequality (114), let y be a variation in the class Y and construct, the variation $\bar{y}$ from Y as in (170). Then by applying (63) to the transformed problem we obtain

$$(171) \quad 0 \le [(\bar{p}_i(t^c)X^{ic}_{b^\sigma b^\tau})\Big|_{c=0}^{c=1} + \mathcal{G}_{b^\sigma b^\tau} + \bar{K}^\alpha\psi^\alpha_{x^ix^j}(t^0)X^{i0}_{b^\alpha}X^{j0}_{b^\tau}]z^\sigma z^\tau$$
$$- \int_{t^0}^{t^1} [\bar{H}_{x^ix^j}y^iy^j + 2\bar{H}_{x^iu^s}y^i\bar{v}^s + \bar{H}_{u^hu^k}\bar{v}^h\bar{v}^k]dt$$

$$\alpha = 1,\dots,m'; \qquad i,j = 1,\dots,N; \qquad \sigma,\tau = 1,\dots,r; \qquad h,k = 1,\dots,K + L'$$
$$s = 1,\dots,K$$

where in (171) we have interchanged components of $\bar{y}$ and y which are equal, and also have recognized from (121) that the function $\bar{H}$ satisfies

$$(172) \qquad \bar{H}_{x^iu^{k+\eta}} = 0 \qquad \eta = 1,\dots,L'$$

Furthermore according to the definition of G and $\bar{G}$ we have

$$(173)\quad \bar{G}_{b^\sigma b^\tau} = G_{b^\sigma b^\tau} + K^\beta \psi^\beta_{x^i x^j}(t^0) X^{i0}_{b^\sigma} X^{j0}_{b^\tau} + \lambda_{p+N+\beta}\left[\psi^\beta_{x^i x^j}(t^s) X^{is}_{b^\sigma} X^{js}_{b^\tau} + \psi^\beta_{x^i}(t^s) X^{is}_{b^\sigma b^\tau}\right]_{s=0}^{s=1}$$

$$\beta = m' + 1,\dots,m; \qquad \sigma,\tau = 1,\dots,r; \qquad i,j = 1,\dots,N$$

Now, using the relations (121) again together with (173) and differentiating the relation for $\bar{H}$ in (121) we re-write (171) as

$$(174)\quad 0 \le \left[(p_i(t^c) - \lambda_{p+N+\beta}\psi^\beta_{x^i}(t^c)) X^{ic}_{b^\sigma b^\tau} + \lambda_{p+N+\beta}(\psi^\beta_{x^i}(t^c) X^{ic}_{b^\sigma b^\tau} + \psi^\beta_{x^i x^j}(t^c) X^{ic}_{b^\sigma} X^{jc}_{b^\tau})\right] z^\sigma z^\tau \Big|_{c=0}^{c=1}$$

$$+ [K^\alpha \psi^\alpha_{x^i x^j}(t^0) X^{i0}_{b^\sigma} X^{j0}_{b^\tau} + G_{b^\sigma b^\tau}] z^\sigma z^\tau - \int_{t^0}^{t^1} [H_{x^i x^j} y^i y^j + 2H_{x^i u^h} y^i v^h + H_{u^h u^k} v^h v^k] dt$$

$$- \lambda_{p+N+\beta} \int_{t^0}^{t^1} [(\psi^\beta_{tx^i x^j} + \psi^\beta_{x^\ell x^i x^j} f^\ell + 2\psi^\beta_{x^\ell x^i} f^\ell_{x^j}) y^i y^j + 2\psi^\beta_{x^\ell x^i} f^\ell_{u^h} y^i v^h$$

$$- 2h_s(t)(\bar{v}^{K+s})^2]dt \qquad h,k = 1,\dots,K; \qquad \beta = m' + 1,\dots,m;$$

$$s = 1,\dots,L'; \qquad \ell,j,i = 1,\dots,N; \qquad \sigma,\tau = 1,\dots,r \qquad \alpha = 1,\dots,m$$

where all derivatives are along a_0.

By the construction (170) together with property (113g) and the properties of the multipliers $h_\eta(t)$ as listed in Theorem 9.1 of [2] we see that the last term in the integrand of (179) vanishes.

Furthermore, recognizing that

$$(175a)\quad \frac{d}{dt}(\psi^\beta_{x^i x^j}) = \psi^\beta_{tx^i x^j} + \psi^\beta_{x^\ell x^i x^j} f^\ell$$

and

(175b) $\psi^{\beta}_{x^{\ell}x^{i}}(f^{\ell}_{x^{j}}y^{j} + f^{\ell}_{u^{h}}v^{h}) = \psi^{\beta}_{x^{\ell}x^{i}}\dot{y}^{\ell}$ $i,j,\ell = 1,\ldots,N;$ $h = 1,\ldots,K$

$\beta = m' + 1,\ldots,m$

we can write the last integral in (174) as

$$(176)\quad -\lambda_{p+N+\beta}\int_{t^0}^{t^1}[\frac{d}{dt}(\psi^{\beta}_{x^i x^j})y^i y^j + 2\psi^{\beta}_{x^i x^j}y^i\dot{y}^j]dt = -\lambda_{p+N+\beta}\int_{t^0}^{t^1}\frac{d}{dt}(\psi^{\alpha}_{x^i x^j}y^i y^j)dt$$

$$= -\lambda_{P+N+\beta}[\psi^{\beta}_{x^i x^j}(t^c)X^{ic}_{b^{\sigma}}X^{jc}_{b^{\tau}}]_{c=0}^{c=1} z^{\sigma}z^{\tau} \qquad \beta = m'+1,\ldots,m; \qquad i,j = 1,\ldots,N;$$

$\sigma,\tau = 1,\ldots,r$

where in the last equality of (176) we have used the property (113k) of the variation y .

Then by putting together (176) and (174) we obtain the inequality (114), thus proving the lemma and Theorem 11.1.

15. A Final Generalization

Our final generalization is to consider the problem of section 11 in which the functions L_{γ}, f^i, ψ^{α}, θ^{η}, include b as an arguement. Thus consider the problem of section 11 with these modifications. Then with a as the arc

(177) a: $x^i(t)$ $u^k(t)$ b^{σ} $t^0 \leq t \leq t^1$ $i = 1,\ldots,N$

$k = 1,\ldots,K$ $\sigma = 1,\ldots,r$

we wish to minimize the functional

$$(178)\qquad I_0(a) = g_0(b) + \int_{t^0}^{t^1} L_0(t,x,u,b)dt$$

among arcs lying in a region R in **txub** space and satisfying the conditions

(179a) $$\dot{x}^i = f^i(t,x,u,b) \qquad i = 1,\ldots,N$$

(179b) $$\psi^\alpha(t,x,b) \le 0 \qquad \alpha = 1,\ldots,m' \qquad \psi^\alpha(t,x,b) = 0 \qquad \alpha = m' + 1,\ldots,m$$

(179c) $$\theta^\eta(t,x,u,b) \le 0 \qquad \eta = 1,\ldots,L' \qquad \theta^\eta(t,x,u,b) = 0 \qquad \eta = L' + 1,\ldots,L$$

(179d) $$I_\gamma(a) \le 0 \qquad \gamma = 1,\ldots,p' \qquad I_\gamma(a) = 0 \qquad \gamma = p' + 1,\ldots,p$$

(179e) $$x^i(t^s) = X^{is}(b) \qquad i = 1,\ldots,N; \qquad s = 0,1$$

where:

(180) $$I_\gamma(a) = g_\gamma(b) + \int_{t^0}^{t^1} L_\gamma(t,x,u,b)dt \qquad \gamma = 1,\ldots,p$$

We assume the same continuity properties and rank properties of gradients with respect to x and u of all our functions as done in section 11 but using the terms of Theorem 11.1 of [2].

We assume again that an arc

(181) $$a_0: \qquad x_0(t) \qquad u_0(t) \qquad b_0 \qquad t^0 \le t \le t^1$$

with $u_0(t)$ continuous is a solution to this problem.

Define normality for this problem in a similar manner to that used in section 11. Thus define the $p + 2N$ functionals, their variations, the indices γ_k $(k = 1,\ldots,\bar{p})$, the constant Ω and the variations $\delta_\omega a$ of a_0 $(\omega = 1,\ldots,\Omega)$ in an analagous fashion to that of section 11. We note now that the variations in our functions include contributions due to variations in b. Then we have for any variation δa, that

$$\delta f^i = f^i_{x^j}\delta x^j + f^i_{u^k}\delta u^k + f^i_{b^\sigma}\, b^\sigma$$

(182)
$$\delta\psi^\alpha = \psi^\alpha_{x^i}\delta x^i + \psi^\alpha_{b^\sigma}\delta b^\sigma$$

$$\delta\theta^\eta = \theta^\eta_{x^i}\delta x^i + \theta^\eta_{u^k}\delta u^k + \theta^\eta_{b^\sigma}\, b^\sigma$$

$$\delta L_\gamma = L_{\gamma_{x^i}}\delta x^i + L_{\gamma_{u^k}}\delta u^k + L_{\gamma_{b^\sigma}}\delta b^\sigma$$

$$\gamma = 0,1,\ldots,p \qquad j,i = 1,\ldots,N \qquad k = 1,\ldots,K \qquad \sigma = 1,\ldots,r$$

$$\alpha = 1,\ldots,m \qquad \eta = 1,\ldots,L$$

Let $\mu_\alpha(t)$, $\lambda_{p+N+r+\alpha}$ be the terms of Theorem 11.1 of [2]. Then for normality of this problem we require the existence of Ω variations $\delta_\omega a$ $(\omega = 1,\ldots,\Omega)$ satisfying the conditions

$$\delta_\omega \dot{x}^i = \delta_\omega f^i \qquad i = 1,\ldots,N$$

$$\delta_\omega\psi^\alpha(t^0) = 0 \text{ if } \mu_\alpha(t^0) \neq \lambda_{p+N+r+\alpha} \qquad \alpha = 1,\ldots,m'$$

$$\delta_\omega\psi^\alpha(t) = 0 \text{ on a neighborhood of } S^\alpha \quad \alpha = 1,\ldots,m'$$

$$\delta_\omega\psi^\alpha(t) = 0 \qquad t^0 \leq t \leq t^1 \qquad \alpha = m' + 1,\ldots,m$$

$$\delta_\omega\theta^\eta(t) = 0 \text{ on a neighborhood of } p^\eta \quad \eta = 1,\ldots,L'$$

$$\delta_\omega\theta^\eta(t) = 0 \qquad t^0 \leq t \leq t^1 \qquad \eta = L' + 1,\ldots,L$$

(where S^α, p^η have meanings here analagous to those of section 11) and also such that the matrix

$$(184) \qquad (J'_\rho(a_0, \delta_\omega a)) \qquad \rho = 1,\ldots,p + 2N \qquad \rho \neq \gamma_k \qquad \omega = 1,\ldots,\Omega$$

is non-singular.

We also define the class Y of variations y

$$(185) \quad y: \qquad y^i(t) \qquad v^k(t) \qquad z^\sigma \qquad t^0 \leq t \leq t^1 \qquad i = 1,\ldots,N$$

$$k = 1,\ldots,K \qquad \sigma = 1,\ldots,r$$

which satisfy conditions entirely analagous to (113) but with the definition of function variation used in (182).

The second order theorem to be proven for this problem is then

Theorem 15.1 Let a_0 be a normal solution to the present problem. Then the multipliers of Theorem 11.1 of [2] K^α, λ_ρ, $\mu_\alpha(t)$, $h_\eta(t)$, $p_i(t)$ $\alpha = 1,\ldots,m'$ $\rho = 0,1,\ldots,p + n + r + m$; $\eta = 1,\ldots,L$, $i = 1,\ldots,N$ can be selected with $\lambda_0 = 1$. When so selected they are unique, except for the multipliers $\lambda_{p+N+r+\alpha}$, $\mu_\alpha(t)$, $p_i(t)$ ($\alpha = 1,\ldots,m$; $i = 1,\ldots,N$) which are respectively unique up to the added terms c^α, c^α, $c^\alpha\psi^\alpha_{x^i}(t)$ all with the same arbitrary constants c^α.

Furthermore with H, and G as the functions of Theorem 11.1 of [2]

$$H = p_i(t)f^i - \lambda_0 L_0 - \lambda_\gamma L_\gamma - \mu_\alpha(t)\phi^\alpha - h_\eta(t)\theta^\eta$$

$$G = \lambda_0 g_0 + \lambda_\gamma g_\gamma + \lambda_{p+i}X^{i0} + \lambda_{p+N+\sigma}b^\sigma - \lambda_{p+N+r+\alpha}[\psi^\alpha(t^s,X^s,b)]_{s=0}^{s=1}$$

$$i = 1,\ldots,N; \quad \gamma = 1,\ldots,p; \quad \alpha = 1,\ldots,m; \quad \eta = 1,\ldots,L; \quad \sigma = 1,\ldots,r$$

then for each variation y in the class Y, we have the following inequality

$$(186)\quad 0 \leq [(p_i(t^c)X^{ic}_{b^\sigma b^\tau})\Big|_{c=0}^{c=1} + G_{b^\sigma b^\tau} + K^\alpha(\psi^\alpha_{x^i x^j}(t^0)X^{i0}_{b^\sigma}X^{j0}_{b^\tau} + 2\psi^\alpha_{x^i b^\tau}(t^0)X^{i0}_{b^\sigma} + \psi^\alpha_{b^\sigma b^\tau}(t^0))]z^\sigma z^\tau$$

$$- \int_{t^0}^{t^1}[H_{x^i x^j}y^i y^j + 2H_{x^i u^h}y^i v^h + H_{u^h u^k}v^h v^k + (2H_{x^i b^\sigma}y^i + H_{b^\sigma b^\tau}z^\tau + 2H_{b^\sigma u^h}v^h)z^\sigma]dt$$

$$\sigma,\tau = 1,\ldots,r \qquad i,j = 1,\ldots,N \qquad h,k = 1,\ldots,K \qquad \alpha = 1,\ldots,m$$

16. Transformation of the Problem

We may transform this problem into one of the type of section 11 by a modification to the method used in section 12 of [2]. Thus we introduce the $2r$ additional parameters and r additional state variables

$$(187)\qquad b^{r+1},\ldots,b^{3r} \qquad x^{N+1},\ldots,x^{N+r}$$

and the conditions

$$(188)\qquad \dot{x}^{N+\sigma} = f^{N+\sigma} = 0 \qquad \sigma = 1,\ldots,r$$

$$X^{N+\sigma,0} = b^{2r+\sigma} \qquad X^{N+\sigma,1} = b^{r+\sigma} \qquad \sigma = 1,\ldots,r$$

The variables $x^{N+\sigma}$ replace the variables b^σ as arguments in the functions f^i, L_γ, ψ^α, ϕ^α, θ^η, and we are concerned now with arcs $\bar{a}$

$$(189)\quad \bar{a}: \qquad \bar{x}^j(t) \qquad \bar{u}^k(t) \qquad \bar{b}^\sigma, \qquad t^0 \leq t \leq t^1$$

$$j = 1,\ldots,N+r \qquad k = 1,\ldots,K \qquad \sigma = 1,\ldots,3r$$

and wish to minimize the functional

$$(190)\qquad I_0(a) = g_0(\bar{b}) + \int_{t^0}^{t^1} L_0(t,\bar{x},\bar{u})dt$$

subject to the conditions

$$\dot{\bar{x}}^j = f^j(t,\bar{x},\bar{u}) \qquad j = 1,\ldots,N + r$$

(191) $$\theta^\eta(t,\bar{x},\bar{u}) \le 0 \qquad \eta = 1,\ldots,L' \qquad \theta^\eta(t,\bar{x},\bar{u}) = 0 \qquad \eta = L' + 1,\ldots,L$$

$$\psi^\alpha(t,\bar{x}) \le 0 \qquad \alpha = 1,\ldots,m' \qquad \psi^\alpha(t,\bar{x}) = 0 \qquad \alpha = m' + 1,\ldots,m$$

$$I_\gamma(\bar{a}) \le 0 \qquad \gamma = 1,\ldots,p' \qquad I_\gamma(\bar{a}) = 0 \qquad \gamma = p' + 1,\ldots,p$$

$$I_{p+\sigma}(\bar{a}) = \bar{b}^\sigma - \bar{b}^{2r+\sigma} = 0 \qquad \sigma = 1,\ldots,r$$

$$\bar{x}^j(t^s) = X^{js}(\bar{b}) \qquad j = 1,\ldots,N + r \qquad s = 0,1$$

where: $X^{N+\sigma,s}$ is defined in (188) and

(192) $$I_\gamma(\bar{a}) = g_\gamma(\bar{b}) + \int_{t^0}^{t^1} L_\gamma(t,\bar{x},\bar{u})dt \qquad \gamma = 1,\ldots,p$$

Now given any arc

$$\bar{a}: \qquad \bar{x}^j(t) \qquad \bar{u}^h(t) \qquad \bar{b}^\sigma \qquad t^0 \le t \le t^1$$

$$j = 1,\ldots,N + r, \quad k = 1,\ldots,K\ , \qquad \sigma = 1,\ldots,3r$$

satisfying (191), then the arc

$$a: \qquad x^i(t) = \bar{x}^i(t) \qquad u^h(t) = \bar{u}^h(t) \qquad b^\sigma = \bar{x}^{N+\sigma} = \bar{b}^\sigma = \bar{b}^{r+\sigma} = \bar{b}^{2r+\sigma}$$

$$i = 1,\ldots,N \qquad k = 1,\ldots,K\ , \qquad \sigma = 1,\ldots,r \qquad t^0 \le t \le t^1$$

will satisfy (179) and by reversing this procedure we may construct an arc $\bar{a}$ satisfying (191) from an arc a which satisfies (179). Furthermore we have that

$$I_\gamma(\bar{a}) = I_\gamma(a) \qquad \gamma = 0,\dots,p$$

for arcs $\bar{a}$ and a corresponding as outlined above.

Thus the present problem is equivalent to the original one and the arc $\bar{a}_0$.

$$(193) \quad \bar{a}_0: \qquad \bar{x}_0^i(t) = x_0^i(t) \qquad i = 1,\dots,N \qquad \bar{x}_0^{N+\sigma}(t) = b_0^\sigma \qquad \sigma = 1,\dots,r$$

$$\bar{u}_0^h(t) = u_0^h(t) \qquad h = 1,\dots,K \qquad \bar{b}_0^\sigma = \bar{b}_0^{r+\sigma} = \bar{b}_0^{2r+\sigma} = b_0^\sigma \qquad \sigma = 1,\dots,r$$

(where the unbarred quantities are from the arc a_0 of (181)) will be a solution to the transformed problem if the arc a_0 of (181) is a solution to the original problem.

In this form, the transformed problem is of the type in section 11 and then according to Theorem 9.1 of [2], there are multipliers $\bar{\lambda}_\rho$, $\bar{\mu}_\alpha(t)$, $\bar{h}_\eta(t)$, $\bar{p}_j(t)$, $\bar{K}^\alpha$ $\quad \rho = 0,1,\dots,\tilde{p} + N + r + m$; $\tilde{p} = p + r$; $\alpha = 1,\dots,m$; $\eta = 1,\dots,L$; $j = 1,\dots,N + r$ and functions $\bar{H}$ and $\bar{G}$ of the form:

$$(194) \quad \bar{H} = \bar{p}_i(t)f^i + \bar{p}_{N+\sigma}(t)f^{N+\sigma} - \bar{\lambda}_0 L_0 - \bar{\lambda}_\gamma L_\gamma - \bar{\mu}_\alpha(t)\phi^\alpha - \bar{h}_\eta(t)\theta^\eta$$

$$\bar{G} = \bar{\lambda}_0 g_0 + \bar{\lambda}_\gamma g_\gamma + \bar{\lambda}_{p+\sigma}(\bar{b}^\sigma - \bar{b}^{2r+\sigma}) + \bar{\lambda}_{p+j}X^{j0} - \bar{\lambda}_{\tilde{p}+N+r+\alpha}[\psi^\alpha(t^s,X^s)]_{s=0}^{s=1}$$

$$i = 1,\dots,N; \qquad \sigma = 1,\dots,r; \qquad \gamma = 1,\dots,p; \qquad \alpha = 1,\dots,m;$$

$$\eta = 1,\dots,L \qquad j = 1,\dots,N + r \qquad \tilde{p} = p + r$$

Now we have modified the transformed problem from that of section 12 of [2] and hence also the proof of Theorem 11.1 of [2]. Then since we are using the terms of Theorem 11.1 of [2] in Theorem 15.1 we should show that we can still prove the former theorem under this modification.

We next prove

Lemma 16.1 With the transformed problem defined above, we can prove Theorem 11.1 of [2].

Proof: Upon inspection of the proof of Theorem 11.1 of [2] in section 12 of that paper, it is seen that with the following modifications the proof of that theorem is still valid:

i) y^σ is replaced by $x^{N+\sigma}$

ii) the number of multipliers $\bar{\lambda}$ is increased by r with these additional multipliers acting as coefficients of the additional isoperimetric functions $(\bar{b}^\sigma - \bar{b}^{2r+\sigma})$, with corresponding changes in index notation so that $\bar{p} = \tilde{p} + N$, $\hat{p} = \tilde{p} + N + r$; with $\tilde{p} = p + r$

iii) the relations between the barred multipliers of Theorem 9.1 of [2] and the multipliers of Theorem 11.1 of [2] are(1)

$$\lambda_\gamma = \bar{\lambda}_\gamma \quad (\gamma = 0,\dots,p) \qquad \lambda_{p+\rho} = \bar{\lambda}_{\tilde{p}+\rho} \quad (\rho = 1,\dots,N + r + m)$$

$$K^\alpha = \bar{K}^\alpha \qquad \mu_\alpha(t) = \bar{\mu}_\alpha(t) \qquad (\alpha = 1,\dots,m)$$

$$p_i(t) = \bar{p}_i(t) \quad (i = 1,\dots,N) \qquad h_\eta(t) = \bar{h}_\eta(t) \quad (\eta = 1,\dots,L)$$

iv) the function $\bar{G}$ is modified as above

v) condition (101) includes the condition $\bar{\lambda}_\gamma = 0$ $(0 \le \gamma \le \tilde{p})$ instead of $\lambda_\gamma = 0 (0 \le \gamma \le \tilde{p})$

(1) Recall that we omit terms corresponding to the variable time aspects of problems in [2].

vi) the relation (105-1) is replaced by

(195a) $$\bar\lambda_0\bar\delta g_0 + \bar\lambda_\gamma\bar\delta g_\gamma + \bar\lambda_{p+\sigma}\bar\delta\bar b^\sigma + \bar\lambda_{\tilde p+i}\bar\delta X^{i0} - \bar\lambda_{\tilde p+N+r+\alpha}[\psi^\alpha_{x^i}(t^s)\bar\delta X^{is}]^{s=1}_{s=0} + [\bar p_i(t^s)\bar\delta X^{is}]^{s=1}_{s=0} = 0$$

and

(195b) $$[\bar\lambda_{p+N+r+\alpha}\psi^\alpha_{x^{N+\sigma}}(t^0) - \bar\lambda_{p+\sigma} + \bar\lambda_{\tilde p+N+\sigma} - \bar p_{N+\sigma}(t^0)]db^{2r+\sigma} = 0$$

$$\gamma = 1,\dots,p; \quad i = 1,\dots,N; \quad \alpha = 1,\dots,m; \quad \sigma = 1,\dots,r$$

$$\tilde p = p + r$$

where we have separated out the parts of (105-1) that can now be set to zero separately since the terms of (195a) do not depend on $b^{2r+\sigma}$ $(\sigma = 1,\dots,r)$ and those of (195b) are not functions of b^σ $(\sigma = 1,\dots,r)$ and where $\bar\delta\bar b$ refers to any variation in the $\bar b$ vector and for example $\bar\delta X^{i0} = X^{i0}_{b^\tau}\bar\delta\bar b^\tau$ $(\tau = 1,\dots,3r)$.

Then since $\bar\delta\bar b^{2r+\sigma}$ is arbitrary, we must have that

(196) $$\bar\lambda_{p+\sigma} = \bar\lambda_{\tilde p+N+r+\alpha}\psi^\alpha_{x^{N+\sigma}}(t^0) + \bar\lambda_{\tilde p+N+\sigma} - \bar p_{N+\sigma}(t^0)$$

Then by (102) and (107) of [2] together with the above relations between multipliers, and (196) we see that (195a) is equivalent to (93) of [2] proving (93). In order to prove the remaining statement of Theorem 11.1 of [2] we employ the extra statement that we have proven is true in the theorems of [2]. By Theorem 9.1 of [2] applied to the transformed problem, this statement takes the form that if the following are true:

(197a) $$\bar K^\alpha = 0, \quad \bar\lambda_{\tilde p+N+r+\alpha} = \bar\mu_\alpha(\bar t) = s^\alpha \quad (\alpha = 1,\dots,m) \quad \bar p_j(\bar t) = s^\alpha\psi^\alpha_{x^j}(\bar t) \quad j \notin K.$$

$$\bar p_i(t_i) = s^\alpha\psi^\alpha_{x^i}(t_i) \, i \in K, \quad \bar p_{N+\sigma}(t_{N+\sigma}) = s^\alpha\psi^\alpha_{x^{N+\sigma}}(t_{N+\sigma}) \quad \sigma = 1,\dots,r$$

and also $\bar{\lambda}_\gamma = 0$ if $L_\gamma \not\equiv 0$ where K is the set of indices $(1 \le i \le N)$ for which $f^i \equiv 0$, and where s^α $(\alpha = 1,\ldots,m)$ are arbitrary constants, then we must have that

$$\bar{p}_j(t) = s^\alpha \psi^\alpha_{x^j}(t) \qquad t^0 \le t \le t^1 \qquad j = 1,\ldots,N + r \tag{197b}$$

Next, assume that condition (96) of [2] is true so that

$$\lambda_\gamma = 0 \qquad 0 \le \gamma \le p \qquad K^\alpha = 0 \tag{198}$$

$$\lambda_{p+N+r+\alpha} = \mu_\alpha(\bar{t}) = s^\alpha \quad \alpha = 1,\ldots,m \quad p_j(\bar{t}) = s^\alpha \psi^\alpha_{x^j}(\bar{t}) \ j \notin K$$

$$p_i(t_i) = s^\alpha \psi^\alpha_{x^i}(t_i) \quad i \in K.$$

for constants s^α $(\alpha = 1,\ldots,m)$ and points $\bar{t}$, t_i in $[t^0,t^1]$ and with terms above, coming from Theorem 11.1 of [2]. Then according to the relations between the multipliers as indicated above (195) together with (106) of [2] we see that the conditions stated above (197b) hold so that (197b) holds. Using (196) this means that for the present situation

$$\bar{\lambda}_{p+\sigma} = s^\alpha \psi^\alpha_{x^{N+\sigma}}(t^0) + \bar{K}^\alpha \psi^\alpha_{x^{N+\sigma}}(t^0) - s^\alpha \psi^\alpha_{x^{N+\sigma}}(t^0) = 0 \qquad \sigma = 1,\ldots,r \tag{199}$$

In (199) we have used (197b) and the last relation of (94) of [2] together with the relations between the multipliers as indicated above (195) of [2]. Since these extra r multipliers were the only part of (101) of [2] which had to be proven, then (96) of [2] implies (101) of [2] so that (96) of [2] and hence Theorem 11.1 of [2] and Lemma 16.1 are proven.(1)

(1) Notice that we do not establish an "extra statement" in Theorem 11.1 of [2] since the purpose of that statement has here been served.

Now define a variation for the transformed problem as

$$\bar{\bar{\delta}}\bar{a}: \quad \bar{\bar{\delta}}\bar{x}^j(t) \quad \bar{\bar{\delta}}\bar{u}^k(t) \quad \bar{\bar{\delta}}\bar{b}^\sigma \quad t^0 \le t \le t^1$$

$$j = 1,\ldots,N+r \quad k = 1,\ldots,K \quad \sigma = 1,\ldots,3r$$

We next show that the transformed problem is normal as follows:

Lemma 16.2 Normality of the problem of section 15 implies normality of the transformed problem.

<u>Proof</u>: Using the variations $\delta_\omega a$ $(\omega = 1,\ldots,\Omega)$ defined above (183), then define the Ω variations $\bar{\delta}_\omega\bar{a}$ of $\bar{a}_0$ as:

$$(200a) \quad \bar{\delta}_\omega\bar{a}: \quad \bar{\delta}_\omega\bar{x}^i = \delta_\omega x^i \quad i = 1,\ldots,N \quad \bar{\delta}_\omega\bar{u}^k = \delta_\omega u^k \quad k = 1,\ldots,K$$

$$\bar{\delta}_\omega\bar{x}^{N+\sigma} = \bar{\delta}_\omega\bar{b}^\sigma = \bar{\delta}_\omega\bar{b}^{r+\sigma} = \bar{\delta}_\omega\bar{b}^{2r+\sigma} = \delta_\omega b^\sigma \quad \sigma = 1,\ldots,r$$

(where the unbarred terms come from the variations $\delta_\omega a$ and Ω is the constant of section 15.)

We also construct $2r$ additional variations $\bar{\delta}_{\Omega+\rho}\bar{a}$ and $\bar{\delta}_{\Omega+r+\rho}\bar{a}$ $(\rho = 1,\ldots,r)$ where for the variation $\bar{\delta}_{\Omega+r+\rho}\bar{a}$ only the terms $\bar{\bar{\delta}}\bar{b}^{2r+\rho}$ and $\bar{\bar{\delta}}\bar{b}^\rho$ are non zero and each of these is set to one, while for the variation $\bar{\delta}_{\Omega+\rho}\bar{a}$ only the term $\bar{\bar{\delta}}\bar{b}^{r+\rho}$ is non zero and is set equal to one. Thus

$$(200b) \quad \bar{\delta}_{\Omega+\rho}\bar{a}: \quad \bar{\delta}_{\Omega+\rho}\bar{x}^j = 0 \quad \bar{\delta}_{\Omega+\rho}\bar{u}^k = 0 \quad \bar{\delta}_{\Omega+\rho}\bar{b}^\sigma = 0 \quad \bar{\delta}_{\Omega+\rho}\bar{b}^{r+\sigma} = \delta_{\rho,\sigma} \quad \bar{\delta}_{\Omega+\rho}\bar{b}^{2r+\sigma} = 0$$

$$\bar{\delta}_{\Omega+r+\rho}\bar{a}: \quad \bar{\delta}_{\Omega+r+\rho}\bar{x}^j = 0 \quad \bar{\delta}_{\Omega+r+\rho}\bar{u}^k = 0 \quad \bar{\delta}_{\Omega+r+\rho}\bar{b}^\sigma = \delta_{\rho,\sigma} \quad \bar{\delta}_{\Omega+r+\rho}\bar{b}^{r+\sigma} = 0$$

$$\bar{\delta}_{\Omega+r+\rho}\bar{b}^{2r+\sigma} = \delta_{\rho,\sigma}$$

$$j = 1,\ldots,N + r; \quad k = 1,\ldots,K; \quad \rho,\sigma = 1,\ldots,r$$

Finally, construct r more variations $\bar{\delta}_{\Omega+2r+\rho}\bar{a}$ $(\rho = 1,\ldots,r)$ such that for the $\rho^{\underline{th}}$ of these variations only $\bar{\delta}\bar{b}^{2r+\rho}$ is non zero and this term is set equal to one. Thus[1]

$$\text{(200c)} \quad \bar{\delta}_{\Omega+2r+\rho}\bar{a}: \quad \bar{\delta}_{\Omega+2r+\rho}\bar{x}^j = 0 \quad \bar{\delta}_{\Omega+2r+\rho}\bar{u}^k = 0 \quad \bar{\delta}_{\Omega+2r+\rho}\bar{b}^\sigma = \bar{\delta}_{\Omega+2r+\rho}\bar{b}^{r+\sigma} = 0$$

$$\bar{\delta}_{\Omega+2r+\rho}\bar{b}^{2r+\sigma} = \delta_{\rho\sigma} \qquad j = 1,\ldots,N + r; \; k = 1,\ldots,K; \; \sigma,\rho = 1,\ldots,r$$

The need for all of these variations will now become evident, for corresponding to the functionals of section 11, introduce the functionals here as

$$\text{(201)} \qquad \bar{J}_\gamma(\bar{a}) = I_\gamma(\bar{a}) \qquad \gamma = 0,1,\ldots,p$$

$$\bar{J}_{p+\sigma}(\bar{a}) = \bar{b}^\sigma - \bar{b}^{2r+\sigma} \qquad \sigma = 1,\ldots,r$$

$$\bar{J}_{\tilde{p}+i}(\bar{a}) = \bar{x}^i(t^1) - X^{i1}(\bar{b}) \qquad i = 1,\ldots,N$$

$$\bar{J}_{p+N+\sigma}(\bar{a}) = \bar{x}^{N+\sigma}(t^1) - \bar{b}^{r+\sigma} \qquad \sigma = 1,\ldots,r$$

$$\bar{J}_{\tilde{p}+N+r+i}(\bar{a}) = X^{i0}(\bar{b}) - \bar{x}^i(t^0) \qquad i = 1,\ldots,N$$

$$\bar{J}_{\tilde{p}+2N+r+\sigma}(\bar{a}) = \bar{b}^{2r+\sigma} - \bar{x}^{N+\sigma}(t^0) \qquad \sigma = 1,\ldots,r \qquad \tilde{p} = p + r$$

where we have used the definition of $X^{n+\sigma,s}$ $\quad \sigma = 1,\ldots,r$; $s = 0,1$.

(1) As before, the term δ with double subscript (as e.g., $\delta_{\rho\sigma}$) denotes the Kronecker Delta

The variations of these functionals due to a variation $\overline{\delta a}$ introduced above Lemma 15.2 are then:

$$(202)\qquad \bar{J}'_{\gamma}(\bar{a}_0,\overline{\delta a}) = I'_{\gamma}(\bar{a}_0,\overline{\delta a}) \qquad \gamma = 0,1,\ldots,p$$

$$\bar{J}'_{p+\sigma}(\bar{a}_0,\overline{\delta a}) = \overline{\delta b}^{\sigma} - \overline{\delta b}^{2r+\sigma} \qquad \sigma = 1,\ldots,r$$

$$\bar{J}'_{\tilde{p+i}}(\bar{a}_0,\overline{\delta a}) = \overline{\delta x}^{i}(t^1) - \bar{\delta}X^{i1} \qquad i = 1,\ldots,N$$

$$\bar{J}'_{\widetilde{p+N+\sigma}}(\bar{a}_0,\overline{\delta a}) = \overline{\delta x}^{N+\sigma}(t^1) - \overline{\delta b}^{r+\sigma} \qquad \sigma = 1,\ldots,r$$

$$\bar{J}'_{\widetilde{p+N+r+i}}(\bar{a}_0,\overline{\delta a}) = \bar{\delta}X^{i0} - \overline{\delta x}^{i}(t^0) \qquad i = 1,\ldots,N$$

$$\bar{J}'_{\widetilde{p+2N+r+\sigma}}(\bar{a}_0,\overline{\delta a}) = \overline{\delta b}^{2r+\sigma} - \overline{\delta x}^{N+\sigma}(t^0) \qquad \sigma = 1,\ldots,r$$

By the construction (200) together with the properties (183) of the variations $\delta_{\omega} a$ $(\omega = 1,\ldots,\Omega)$ and the relationships between the multipliers as indicated above (195) we see that the variations $\bar{\delta}_{\varpi}\bar{a}$ $(\varpi = 1,\ldots,\Omega+3r)$ defined above satisfy the conditions corresponding to (112a) through (112f) for the transformed problem. Furthermore, by our construction of the arc $\bar{a}_0$ and the definition of the functionals (201) and the functionals $J_{\rho}(a)$ $(\rho = 0,1,\ldots,p+2N)$ of section 15 we see that

$$(203)\qquad \bar{J}_{\gamma}(\bar{a}_0) = J_{\gamma}(a_0) \qquad \gamma = 0,1,\ldots,p$$

$$\bar{J}_{\tilde{p+i}}(\bar{a}_0) = J_{p+i}(a_0) \qquad i = 1,\ldots,N$$

$$\bar{J}_{\widetilde{p+N+r+i}}(\bar{a}_0) = J_{p+N+i}(a_0) \qquad i = 1,\ldots,N$$

so that we have in (201) essentially only added 3r more functions $\bar{J}_{p+\sigma}$, $\bar{J}_{\tilde{p}+N+\sigma}$ and $\bar{J}_{\tilde{p}+2N+r+\sigma}$ $(\sigma = 1,\dots,r)$.

Using the property (184) of the variations $\delta_\omega a$ then we see that the matrix

(204) $$(\bar{J}'_\rho(\bar{a}_0,\bar{\delta}_\omega\bar{a}) \qquad \rho = 0,1,\dots,\tilde{p}+2(N+r) \qquad \rho \neq \gamma_k \qquad \rho \neq p+\sigma$$

$$\rho \neq \tilde{p}+N+\sigma \qquad \rho \neq \tilde{p}+2N+r+\sigma \qquad \tilde{p} = p+r \qquad \sigma = 1,\dots,r;$$

$$\omega = 1,\dots,\Omega$$

(where γ_k the indices defined for the transformed problem as in section 11 are the same indices as defined in section 15)is non singular and that the variations of the other functionals of (201) vanish, namely

(205) $$\bar{J}'_{p+\sigma}(\bar{a}_0,\bar{\delta}_\omega\bar{a}) = 0$$

$$\bar{J}'_{\tilde{p}+N+\sigma}(\bar{a}_0,\bar{\delta}_\omega\bar{a}) = 0$$

$$\bar{J}'_{\tilde{p}+2N+r+\sigma}(\bar{a}_0,\bar{\delta}_\omega\bar{a}) = 0 \qquad \sigma = 1,\dots,r \qquad \omega = 1,\dots,\Omega$$

Furthermore the 2r additional variations of (200b) satisfy

(206) $$\bar{J}'_{p+\sigma}(\bar{a}_0,\bar{\delta}_{\Omega+\tau}\bar{a}) = 0$$

$$\bar{J}'_{\tilde{p}+N+\sigma}(\bar{a}_0,\bar{\delta}_{\Omega+\tau}\bar{a}) = -\delta_{\sigma\tau}$$

$$\bar{J}'_{\tilde{p}+2N+r+\sigma}(\bar{a}_0,\bar{\delta}_{\Omega+\tau}\bar{a}) = \delta_{\tau-r,\sigma} \qquad \sigma = 1,\dots,r \qquad \tau = 1,\dots,2r \qquad \tilde{p} = p -$$

Finally, the r variations of (200c) satisfy

(207) $$\bar{J}'_{p+\sigma}(\bar{a}_0, \bar{\delta}_{\Omega+2r+\rho}\bar{a}) = \delta_{\sigma\rho} \qquad \rho,\sigma = 1,\ldots,r$$

Then using all the $\Omega + 3r$ variations of (200) we see that the matrix

(208) $$(\bar{J}'_\rho(\bar{a}_0, \bar{\delta}_\omega \bar{a})) \qquad \rho = 0,1,\ldots,\tilde{p} + 2(N+r) \qquad \rho \neq \gamma_k$$

$$\omega = 1,\ldots,\bar{\Omega} = \Omega + 3r$$

is non-singular. Thus according to section 11, the transformed problem is normal and the lemma is proven.

Next define the class $\bar{Y}$ of variations $\bar{y}$

(209) $$\bar{y}: \qquad \bar{y}^j(t) \qquad \bar{v}^k(t) \qquad \bar{z}^\sigma \qquad t^0 \leq t \leq t^1$$

$$j = 1,\ldots,N+r \qquad k = 1,\ldots,K \qquad \sigma = 1,\ldots,3r$$

which satisfy conditions analagous to (113)

(210) $$\dot{\bar{y}}^i = \bar{\delta}_{\bar{y}} f^i \qquad i = 1,\ldots,N \qquad \dot{\bar{y}}^{N+\sigma} = \bar{\delta}_{\bar{y}} f^{N+\sigma} = 0 \qquad \sigma = 1,\ldots,r$$

$$\bar{\delta}_{\bar{y}}\psi^\alpha(t) \leq 0 \text{ on a neighborhood of } S^\alpha \qquad \alpha = 1,\ldots,m'$$

$$\bar{\delta}_{\bar{y}}\psi^\alpha(t^c) = 0 \text{ if } \bar{\mu}_\alpha(t^c) \neq \bar{\lambda}_{\tilde{p}+N+r+\alpha} \qquad c = 0,1; \qquad \alpha = 1,\ldots,m'$$

$$\bar{\delta}_{\bar{y}}\psi^\alpha(\bar{t}) = 0 \text{ if } \bar{\mu}_\alpha(t) \text{ is not constant near } t = \bar{t} \qquad \alpha = 1,\ldots,m'$$

$$\bar{\delta}_{\bar{y}}\psi^\alpha(t^0) = 0 \text{ if } \psi^\alpha(t^0) = 0 \qquad \alpha = 1,\ldots,m'$$

(210) $\bar{\delta}_{\bar{y}}\psi^{\alpha}(t) = 0 \qquad t^0 \le t \le t^1 \qquad \alpha = m' + 1,\ldots,m$

$\bar{\delta}_{\bar{y}}\theta^{\eta}(t) = 0$ on a neighborhood of $p^{\eta} \qquad \eta = 1,\ldots,L'$

$\bar{\delta}_{\bar{y}}\theta^{\eta}(t) = 0 \qquad t^0 \le t \le t^1 \qquad \eta = L' + 1,\ldots,L$

$\bar{J}'_{\gamma}(\bar{a}_0,\bar{y}) = 0$ if $\bar{\lambda}_{\gamma} \ne 0 \qquad 1 \le \gamma \le p'$

$\bar{J}'_{\gamma}(\bar{a}_0,\bar{y}) \le 0$ if $\bar{\lambda}_{\gamma} = 0 \qquad \gamma \ne \gamma_k \qquad 1 \le \gamma \le p'$

$\bar{J}'_{\rho}(\bar{a}_0,\bar{y}) = 0 \qquad p' < \rho \le \tilde{p} + 2(N+r)$

where: i) $\bar{\mu}_{\alpha}(t)$, $\bar{\lambda}_{\tilde{p}+N+r+\alpha}$, $\bar{\lambda}_{\gamma}$, are terms introduced above (194a) and ii) S^{α}, P^{η} are as defined previously and are the same when determined along either of the arcs a_0 or $\bar{a}_0$.

We next prove

Lemma 16.3 Given a variation y in the class Y of section 15, then we can construct a variation $\bar{y}$ in the class $\bar{Y}$.

Proof: Given the variation y in the class Y

(211) y: $\quad y^i(t) \qquad v^k(t) \qquad z^{\sigma} \qquad t^0 \le t \le t^1$

$i = 1,\ldots,N \qquad k = 1,\ldots,K \qquad \sigma = 1,\ldots,r$

then construct the variation $\bar{y}$ with

(212) $\bar{y}^i(t) = y^i(t) \quad i = 1,\ldots,N \quad \bar{y}^{N+\sigma} = \bar{z}^{\sigma} = \bar{z}^{r+\sigma} = \bar{z}^{2r+\sigma} = z^{\sigma} \quad \sigma = 1,\ldots,r$

$\bar{v}^k(t) = v^k(t) \qquad k = 1,\ldots,K$

Then analagous to the proof in Lemma 16.2, this construction together with the properties of the variation y, the relations between the multipliers as listed above (195) and the definition of the functionals $\bar{J}$, proves that $\bar{y}$ satisfies the conditions (210) and hence is in the class $\bar{Y}$. Thus the Lemma is proven.

17. Proof of Theorem 15.1

We now prove

Lemma 17.1 Theorem 11.1 implies Theorem 15.1

Proof: Applying Theorem 11.1 to the transformed problem and using the relations listed above (195) between the multipliers of Theorems 11.1 and 15.1, we see that the first two statements of Theorem 15.1 follow directly from the first two statements of Theorem 11.1.

In order to establish the inequality (186), let y be in the class Y and by Lemma 16.3 let $\bar{y}$ be the variation constructed from it. Then by applying (114) to the transformed problem, and using (212) we have

$$(213)\quad 0 \le \left[\left(\bar{p}_i(t^c)X^{ic}_{b^\sigma b^\tau}\right)_{c=0}^{c=1} + \bar{G}_{b^\sigma b^\tau} + \bar{K}^\alpha \psi^\alpha_{x^\ell x^s}(t^0)X^{\ell,0}_{b^\sigma}X^{s,0}_{b^\tau}\right]\bar{z}^\sigma \bar{z}^\tau -$$

$$\int_{t^0}^{t^1}\left[\bar{H}_{x^\ell x^s}\bar{y}^\ell \bar{y}^s + 2\bar{H}_{x^\ell u^k}\bar{y}^\ell \bar{v}^k + \bar{H}_{u^h u^k}\bar{v}^h \bar{v}^k\right]dt \qquad i = 1,\dots,N \quad \sigma,\tau = 1,\dots,3r$$

$$\ell, s = 1,\dots,N + r \qquad h,k = 1,\dots,K$$

where in (213) we have recognized that the second derivatives of $X^{N+\sigma,s}$ ($\sigma = 1,\dots,r$; $s = 0,1$) vanish.

Now by: i) the forms of the functions $X^{r,c}$ $r = 1,\ldots,N + r$, $c = 0,1$ (of (179e) and (188)), and G, $\bar{G}$ given respectively in Theorem 15.1 and (194), ii) the relation between the multipliers as indicated above (195), iii) the construction of (212) and iv) recognizing that according to the form of H, $\bar{H}$ given respectively in Theorem 15.1 and (194), differentiation of $\bar{H}$ with respect to $x^{N+\sigma}$ is the same as differentiating H with respect to b^{σ} with a similar statement holding for ψ^{α}, we obtain the inequality (186) from (213). Thus the lemma and hence also Theorem 15.1 are proven.

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This paper is a sequel to an article by the author, concerned with a certain canonical problem in optimal control involving constraints of the type $\psi^{\alpha}(t,x) \le 0 \quad \alpha = 1,\dots,m$. In that article a set of second order conditions necessary for a solution arc was obtained. In this paper those results are extended to a general control problem involving the above type of

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